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ADAPTIVE MESH REFINEMENT TECHNIQUES FOR HIGH-ORDER FINITE-VOLUME WENO SCHEMES

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Abstract. *This paper demonstrates the capabilities of Adaptive Mesh Refinement Techniques (AMR) on 2D hybrid unstructured meshes, for high order finite volume WENO methods. The AMR technique developed is a conformal adapting unstructured hybrid quadrilaterals and triangles (quads & tris) technique for resolving sharp flow features in accurate manner for steady-state and time dependent flow problems. In this method, the mesh can be refined or coarsened which depends on an error estimator, making decision at the parent level whilst maintaining a conformal mesh, the unstructured hybrid mesh refinement is done hierarchically. When a numerical method can work on a fixed conformal mesh this can be applied to do dynamic mesh adaptation. Two Refinement strategies have been devised both following a H-P refinement technique, which can be applied for providing better resolution to strong gradient dominated problems. The AMR algorithm has been tested on cylindrical explosion test and forward facing step problems.*

1 INTRODUCTION

Adaptive mesh refinement (AMR) techniques are well known and vastly used technique for the accurate capturing of the solution features in a steady or an unsteady simulation. The adaptive refinement enables to capture complex solution features by performing refinement in critical areas without having to refine the whole mesh. AMR has become a standard practice in triangular and tetrahedral meshes for various applications, the unique topological properties of these elements allow for local refinement and maintaining good element quality and retaining the conformity of the mesh [6]. For a quadrilateral mesh the general approach of refinement generates non-conformal elements. This non-conformity may allow local refinement but introduces hanging nodes, which requires special augmentation of the PDE solution to deal with these special nodes. Hanging nodes are generally dealt by constraining the solution at these nodes to be dependent on the solution at the nodes of the edge it lies on using constraint equations [4].

The adaptive procedures automatically try to refine, coarsen or relocate the mesh or tries to adjust the solution basis to achieve a specific accuracy in an optimal way. The computations generally begin with a trial solution generated on a coarse mesh which has a lower order basis where the error of this solution is assessed. If this fails to satisfy the required accuracy, adjustments are made to obtain the required solution with minimum effort, where we try to reduce the discretisation error to its required. Adaptive methods have been studied for nearly twenty years now and there are still only a few known optimal strategies and few of the common procedures studied till date include, the local refining or coarsening of a mesh (h-refinement), relocating or moving the nodes in a mesh (r-refinement) and locally varying the polynomial degree (p-refinement).

This paper demonstrates the capabilities of AMR on 2D hybrid unstructured meshes, for high order finite volume WENO methods. The AMR technique developed is a conformal adapting unstructured hybrid (quads & tris) technique for resolving sharp flow features in accurate manner for steady-state and time dependent flow problems. In this method, the mesh can be refined or coarsened which depends on an error estimator, making decision at the parent level whilst maintaining a conformal mesh, the unstructured hybrid mesh refinement is done hierarchically. When a numerical method can work on a fixed conformal mesh this can be applied to do dynamic mesh adaptation [4]. The adaptation strategy devised follows a H-P refinement technique, which can be applied for providing better resolution to strong gradient dominated problems.

2 NEED FOR REFINEMENT

Most of the physical problems that are considered for numerical simulation have features with multiple scales in both space and time, which has been a problem for numerical analysis in attempts to resolve more of the scales with evenly spaced grids which require more computational resources, storage and time to execute, because of this the meshes were pre adapted to known features in the solution. This method works well for steady solutions with well-known locations of the local features within the solution, but for solution features which are not well known, multiple solution or remeshing were needed to get the considerable resolution level which required a complex remeshing code. These problems become absurd when unsteady solutions are considered, a small movement in the solution feature may render extensive remeshing of no use. To overcome this, zonal refinement techniques have also been used which also fails when unrestrained feature movement exists. For the adaptive mesh techniques to be useful for unsteady or steady flows where there is no proper knowledge of the solution, it has to adapt dynamically and automatically as the solution evolves. Using a standard mesh generation code

in this situation will require stopping the solution periodically, remeshing, interpolating and then restarting, to resolve multiple scale unsteady problems a dynamic solution adaptive grid technique has to be developed.
With references to the study made by Joe.F.Thompson[11] on grid generation and adaptive techniques, an adaptive mesh refinement technique has to fulfil a few criteria which can be termed as the goals of adaptive mesh refinement. According to this study amr has to,

- Reduce spatial discretisation error.
- Remove grid dependency of the solution to maximum extent.
- Preserve mesh quality as far as possible.
- The results with adaptation have to be quantifiable.
- If solution is time dependent, the adaptation should be dynamic and has to preserve temporal accuracy.
- Once initial criteria is selected, the adaptive process should continue without any user intervention.
- The adaptive technique should be effective.
- There should be minimal error added to the solution.

3 PREVIOUS WORK

There has been extensive research on the conformal refinement of triangular meshes for adaptive simulations as treatment of these element types are easy [2]. On the contrary for quadrilateral meshes it is common to use non-conformal quad tree type refinement with special treatment for the hanging nodes [1]. There have been only a few researchers describing the coarsening and refinement of quadrilateral mesh and relatively fewer which can also handle hybrid mesh elements and also deal with the dynamic settings.

The best Paper on conformal quadrilateral refinement written by Schneider's [8] discusses the method of refinement based on bisection and trisection of the edges. He implies that the trisection of edge strategy simplifies the algorithm, where the information of refinement is communicated from elements to nodes and the templates for refinement are defined based on the number of marked nodes. The refinement templates are selected to keep the scheme stable, where the quality of the elements do not depreciate with increasing refinement levels. In his paper, uniformly refined quadrilaterals which are trisected have 9 child cells and the templates for the adjacent cells which terminate the refinement have bisected edge. Schneider's schemes are more complicated to implement than the scheme to be presented here, but still is a valid and tested scheme for conformal quadrilateral refinement and has also been used by others like Zhang and Bajaj [12]. Schneider's extended his work to hexahedral but says that certain refinement patterns for the faces of hexahedra may not admit a valid decomposition of the parent hexahedron. Ito et al. [5] has also used the Schneider's approach for octree hexahedral refinement templates.

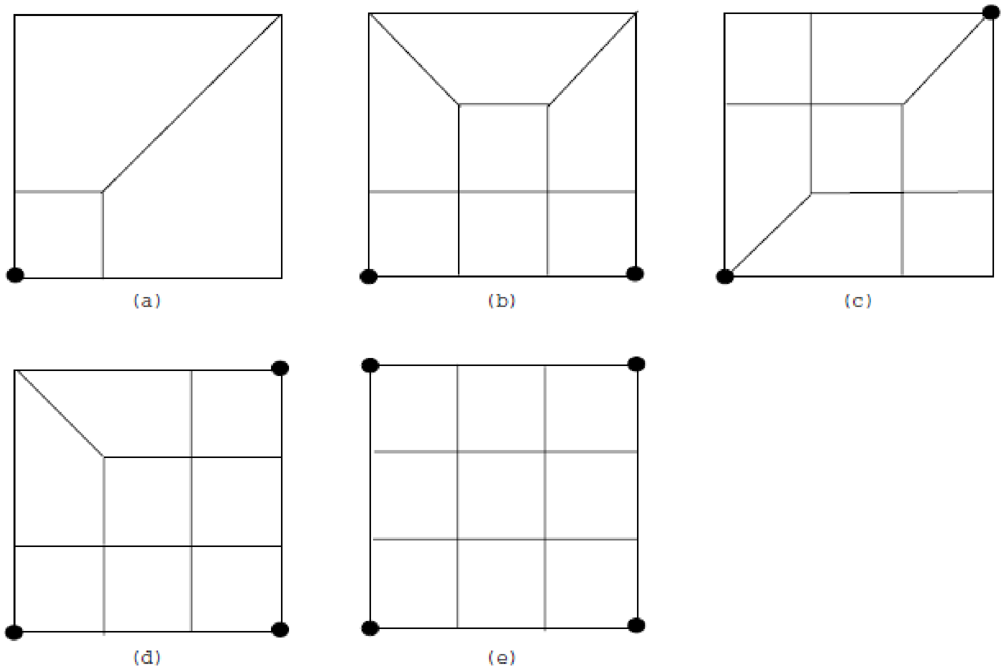


Figure 1. Schniders' subdivision templates showing the trisection and bisection of edges. [8]

Tchon et al. [10 (Tchon, 2004)] has also worked on quadrilateral refinement strategy where they find layers of elements, shrink the layers of elements and reconnect the shrunk layer with the surrounding mesh. This strategy assumes a particular structure to the mesh and a few specific patterns may ignore the issues of multiple levels of refinement, mesh quality and dynamic adaptation which makes this method limited in utility.

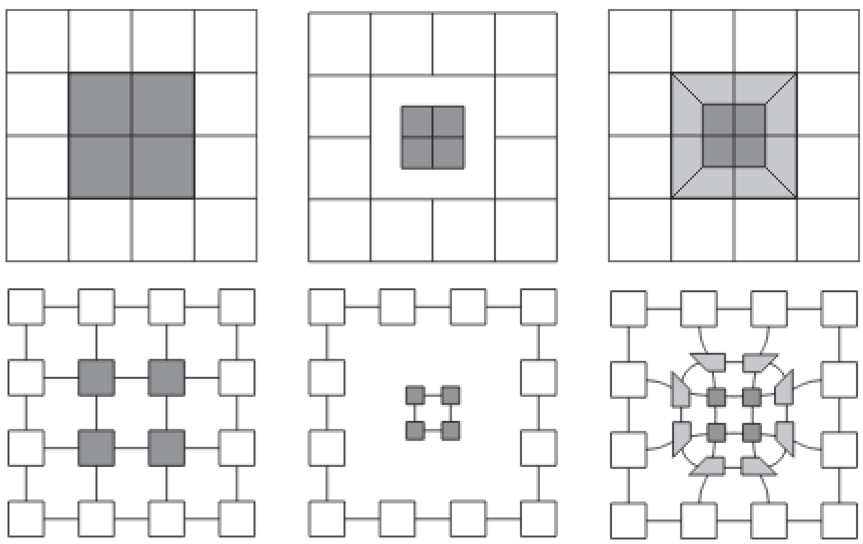


Figure 2. Tchon et al.'s conformal shrink and reconnect method [10]

Benzley et al. [9] proposed quadrilateral mesh coarsening strategies which are general and have an advantage over nested refinement strategies where they can be coarsened beyond the original resolution of the mesh.

In the work presented by Sandhu et al. [7] they use node marking and trisection of edges to define templates for refinement, which is quite similar to the work done by Garimella Rao [4] where he proposes a technique for multilevel adaptive refinement of quadrilateral meshes where the elements are kept conformal all the time. Garimella uses one less defined template than Sandhu et al.. With all these methods into consideration, they show only static refinement and aspects of dynamic adaptation and solution mapping have not been explored.
The research that is closest to the presented work is the paper by Michael Dumbser et al. [3] which talks about high order ader-weno finite volume scheme with AMR. The higher order spatial accuracy is got by using a weno reconstruction and the high order one step time discretisation is obtained through discontinuous galerkin predictor method. The AMR strategy is implemented cell-by-cell with standard tree type algorithm and has also been parallelised. The strategies have been tested on nonlinear systems including the Euler equations and with varying orders of accuracy to show the results of using AMR.

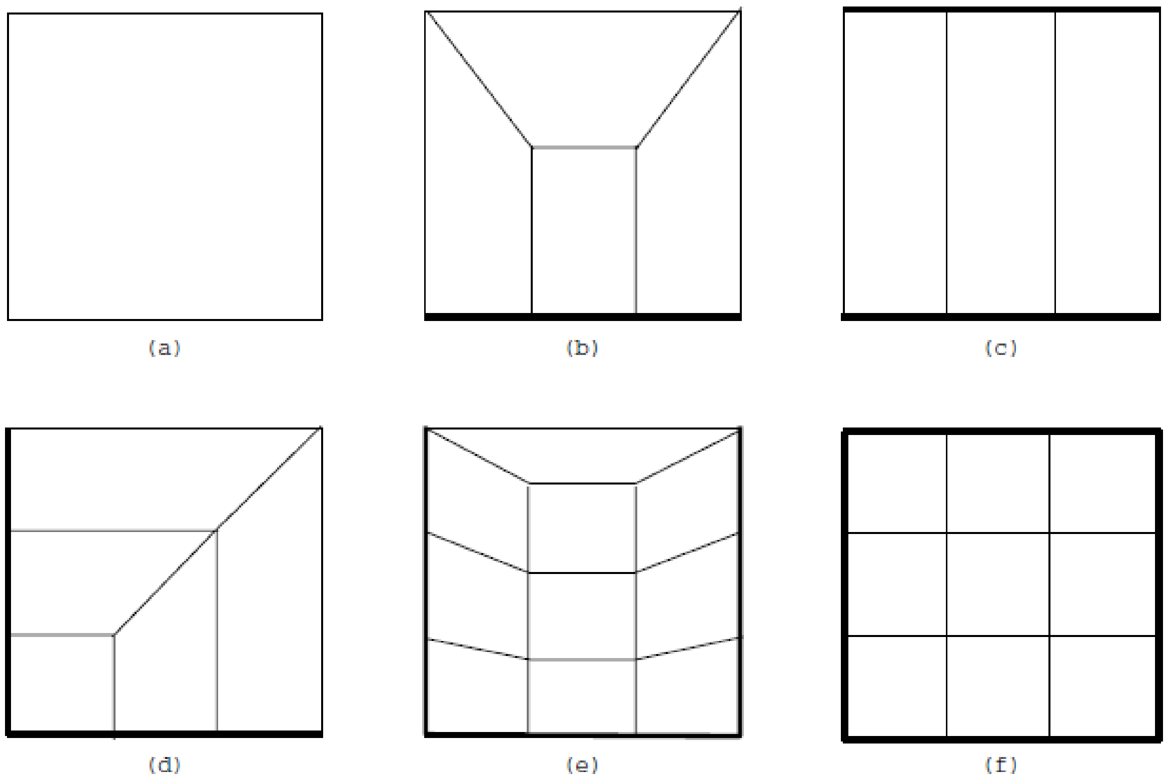


Figure 3. Graimella Rao's subdivision templates demonstrating edge based refinement [4]

4 AMR ALGORITHM

The mesh adaptation technique that has been developed is used in conjunction with the high order unstructured finite volume solver (ucns) which is capable of obtaining 7th order spatial accuracy with weno schemes. The AMR technique implemented is a fully conformal method that can work on hybrid meshes, which follows a hierarchical tree based data structure. The refinement or coarsening is node based and the decision is always taken at the parent level i.e. the data of the initial parent cell is preserved and is always carried over to the number of adaptations done during the process which limits the coarsening to be done only up to the initial mesh level. The main point of this technique is to preserve the order of accuracy with varying gradients and also at points where the solution is smooth using solution transfer/remapping techniques. The other advantage of this method developed here is that the refinement or the coarsening process can be done as local as possible and also can be done per cell basis which gives the user more control over the process and is very essential and would aid in better parallelisation of the algorithm.

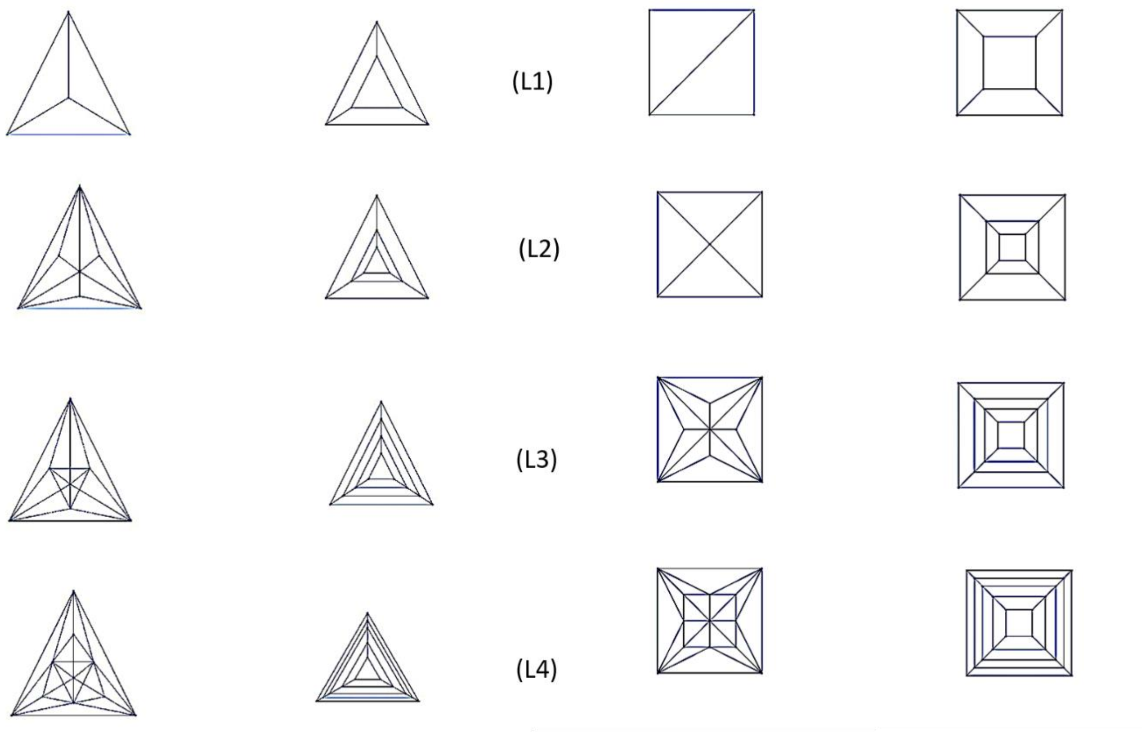


Figure 4. The two strategies of refinement on quadrilateral and triangular cells, showing different levels of refinement.

Two strategies of refinements have been developed for quadrilaterals and triangles which can refine the cells up to 4 levels where the maximum number of children per cell can go up to 21 cells. The above figure shows the subdivision of cells for both the refinement strategies. The subdivision is done based on the nodes and the centre of the cells, where for each level the mid-point is calculated on the diagonals between the end node and the cell centre. This is done repetitively until four levels for the first strategy. The second strategy of subdivision was developed for enabling better stencil marking for the weno schemes to communicate better with the neighbour cells. The subdivision of the cells were kept uniform by following a step by step subdivision where the new cells generated can take up an identical or an averaged area or volume for better solution remapping and transfer.

For the AMR to be activated it is necessary for the solver to start, to get the initial gradients and the solution. The refinement or coarsening is done based on the non dimensionalised gradient levels. With the initial mesh fed to the solver we get the gradients and the solution, the criteria and the levels of refinement are pre-defined in a parameter file based on which gradient to choose. The AMR can be activated in two ways, for every n iterations of the simulation or for every n-th time step of the simulation based on the type of problem being solved. From the initial mesh with the parameters given for refinement the cells are marked, it has to be noted that there can be no coarsening happening at the initial level as the algorithm does not support coarsening of the mesh beyond the initial mesh. As soon as the cells are marked the neighbours of the marked cells as well as the neighbours of the neighbours are marked and only the exclusive neighbour list is populated and is stored for smooth transition of refinement levels which makes sure that the adjacent cell is always only one level different than the target cell. The non dimensionalised gradients are now used to specify the level of refinement to the marked target cells along with their neighbours for a smoother transition. The refinement is done for the marked cells and the data of the parent cells are stored with the children flag and the child cell numbers, this data is used in the next step of adaptation where the decision is taken at a parent level for refinement or coarsening. For the parent cells there is an option of choosing the maximum gradients of the child cell or the average gradients of the cells for obtaining more resolution with adaptation. The solution is remapped on to the cells and it is made sure that the order of accuracy is conserved.

4.1 Solution Remapping & Transfer

Adaptation methods to reduce the solution errors of solving a PDE is highly dependent on the remapping or transfer of quantities from the base mesh to the adapted mesh and while coarsening of the mesh. It is important to remap the solution quantities like the integral quantities such as mass or energy and pointwise quantities such as diffusivity [13]. The remapping of both these quantities have to be done very accurately and more importantly for the integral quantities it has to be conservative. Consider the density of a child, it must be transferred such that the total mass of the parent is conserved. If a group of elements are to be coarsened the integral quantities can be summed up over the children and can be assigned to the parent. With the pointwise quantities the children cells can be averaged weighted by their volume. For conditions of refinement, the mass can be distributed equally over the children or a linear reconstruction of the density can be made over the parent and integrate over the child [13,14]. In the proposed algorithm here, the solution transfer between the refinement levels is treated differently where a volume average is take and remapped to the cells and for refinement the quantities of the parent are mapped as it is to the children cells. Using a summation of masses of the children and passing it to the parent may be a poor choice and might lead to lower order of accuracy.

4.2 Methodology

The AMR algorithm is employed with MUSCL and WENO type of spatial discretisation schemes on hybrid unstructured meshes based on the implementation of [18,19], where they have been applied to a range of inviscid, laminar, transitional and turbulent flows simulations [14-17,20] demonstrating the advantages of high-order schemes in conjunction with hybrid unstructured meshes. The HLLC Riemann solver is used along with an explicit 3_{rd}-order three stage strong stability preserving Runge-Kutta time stepping for advancing the solution in time.

5 RESULTS & DISSCUSSION

The developed AMR is tested on two test cases namely the cylinder explosion problem and the forward facing step. Both the cases demonstrates the capability of dynamic mesh adaptation through time and changing gradients over single element and hybrid mesh domains. The tests are done with three numerical schemes namely muscl second order, weno 3^{rd} order and weno 5^{th} order to study the AMR behavior over increasing order of numerical accuracy.

5.1 Cylinder Explosion Test Case

The two dimensional Euler equation for gas dynamics to solve the considered problem is given as,

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u_x \\ \rho u_y \\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u_x \\ \rho u_x^2+p \\ \rho u_x u_y \\ (E+p)u_x\end{bmatrix}+\frac{\partial}{\partial y}\begin{bmatrix}\rho u_y \\ \rho u_y u_x \\ \rho u_y^2+p \\ (E+p)u_y\end{bmatrix}=0 \qquad (1)$$

Where the pressure p is related to the conserved quantities through the equation of state,

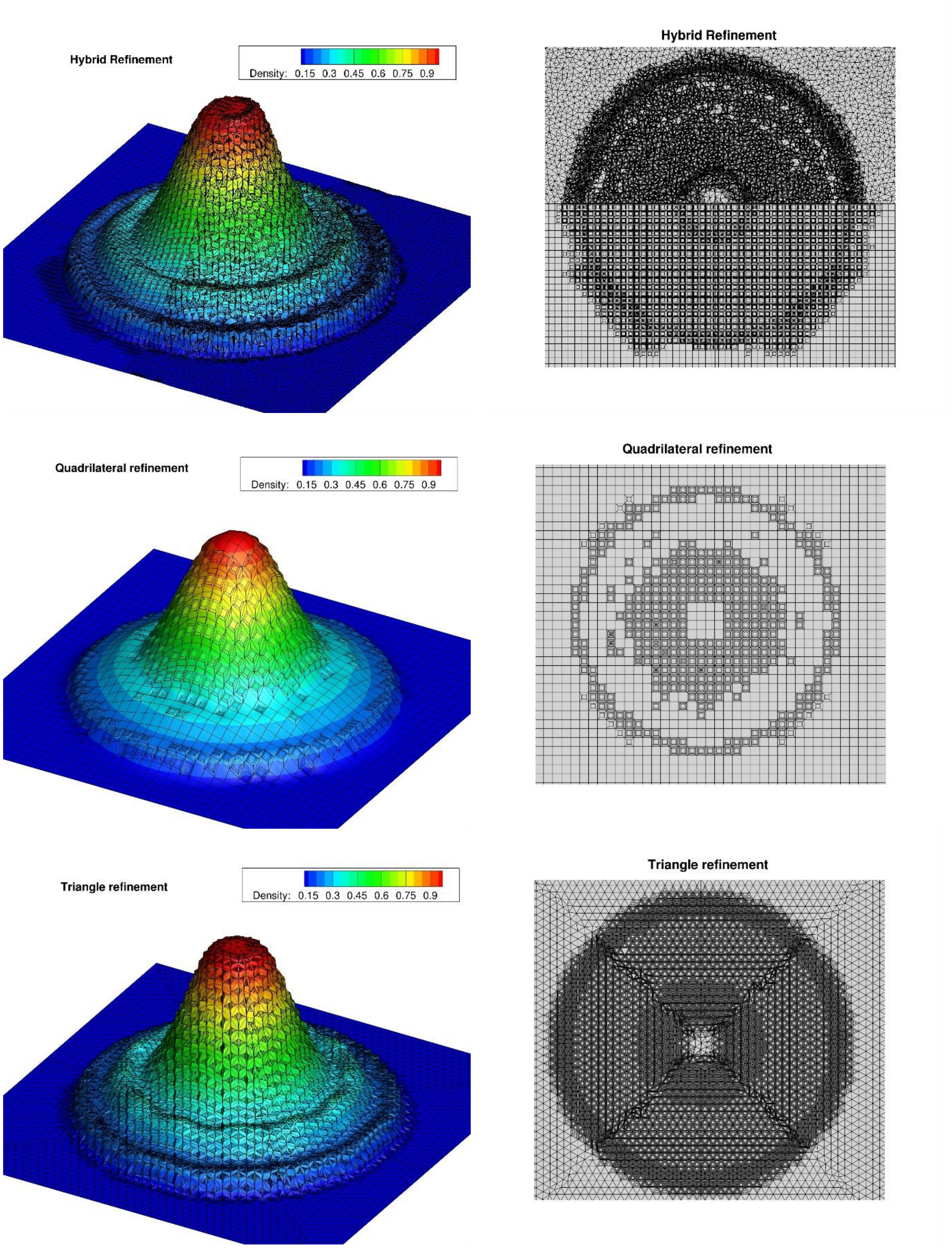
$$p=(\gamma-1)(E-\frac{1}{2}\rho(u_x^2+u_y^2)) \qquad (2)$$

With γ =1.4, which is solved on a square domain in the xy-palne. The initial condition has a circular discontinuity of radius 0.4 centered at (1, 1). The initial data for the problem is defined on a non-dimensional domain of [-1:1] x[-1:1] and has two regions of constant but different values of gas parameters, the initial conditions are,

$$(\rho,p)=\begin{cases}(1.0,1.0), r\le 0.4 \\ (0.125,0.1), r>0.4\end{cases}, u=v, r^2=x^2+y^2 \qquad (3)$$

For the cylinder explosion problem in 2D, a set of three meshes namely a quad mesh with 1600 quadrilaterals, a triangular mesh with 5458 triangle elements and a hybrid mesh with 1711 and 5250 quads and tris respectively are used as the base mesh. These meshes are run on three numerical schemes muscl2, weno3 and weno5 to capture the solution with increasing numerical accuracy. The AMR for all the cases have been set to be moderately aggressive, the simulation is run for a time of two seconds with adaptation set to happen every 0.09 seconds which accounts for 3 adaptations during the course of the simulation. We expect the results to be sharper and where there are sharp gradients and also to conserve the parts where the solution is smooth with varying AMR settings. Refinement levels of l2, l3 and l4 were used for all conditions and the varying levels of non-dimensionalised density was used as the gradient markers for marking the cells for adaptation.

The solution exhibits a circular shock wave propagating away from the center, a circular contact surface travelling in the same direction and a circular rarefaction moving towards the origin. As time evolves it can be observed that a complex wave pattern emerges. The circular shock wave moves outwards and becomes weaker, the contact surface also follows the shock and becomes weaker and at some point in time the contact comes to rest and starts to move inwards. The rarefaction moving towards the center reflects and over expands the flow to create an inward moving shock wave which implodes to the origin, reflects and moves out colliding with the contact surface.



Where

Figure 5. Cylinder explosion with strategy_1 hybrid, quad and tri grid on weno3 scheme, meshes at final time step

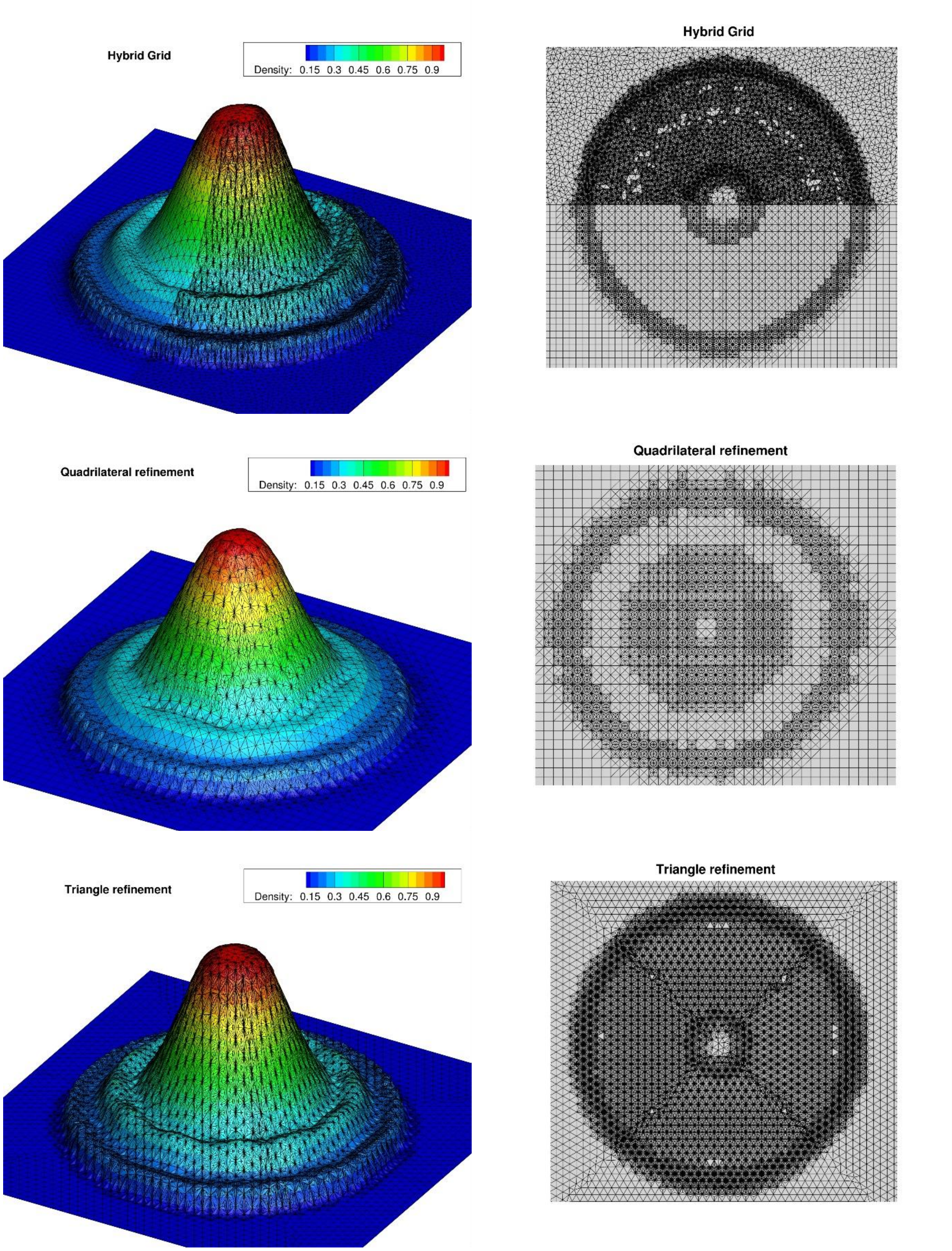


Figure 6.Cylinder explosion with strategy_2 hybrid, quad and tri grid on weno3 scheme, meshes at final time step

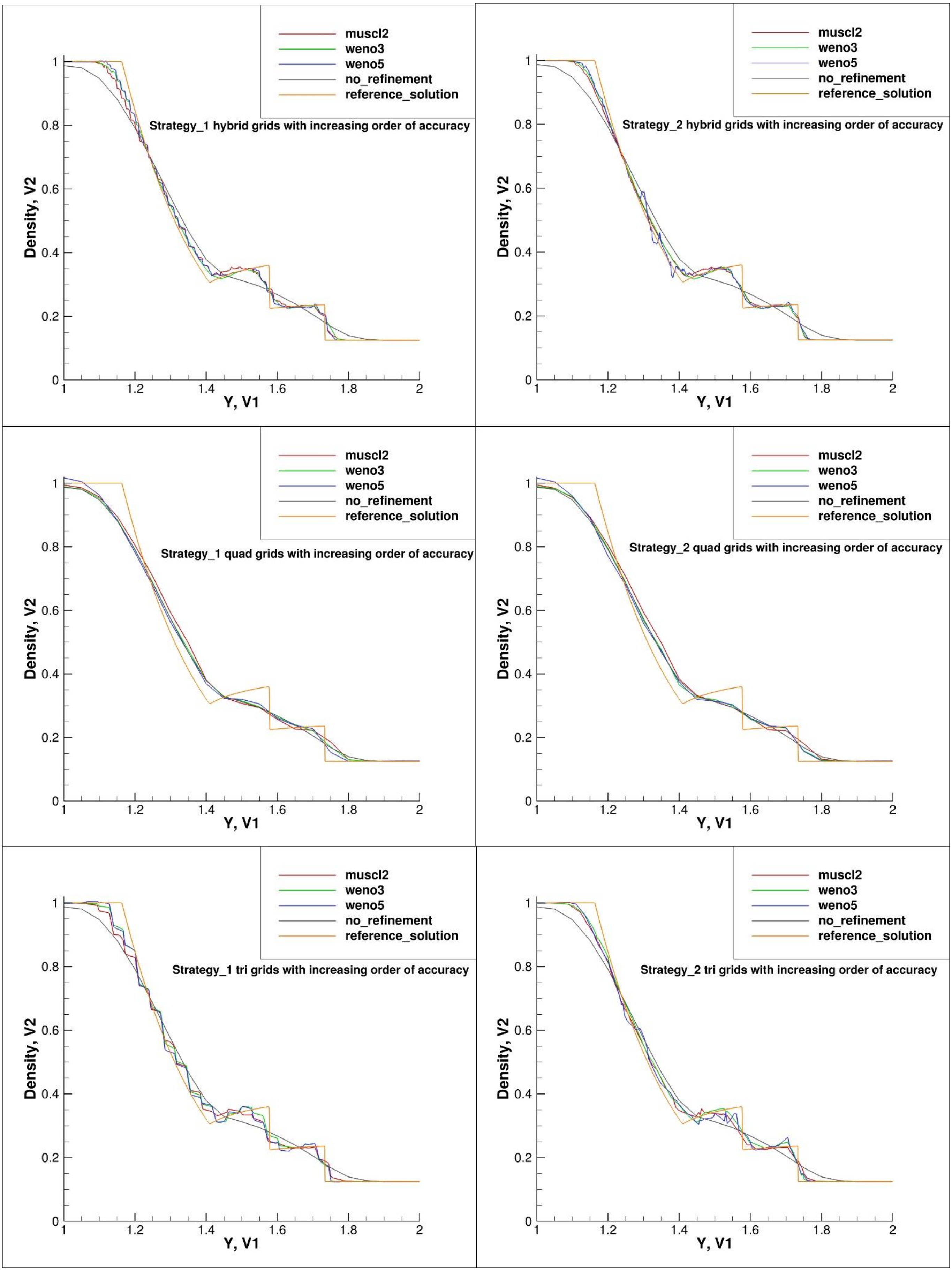


Figure 7.Density plots on different grids with higher order schemes

Figures 5 & 6 show the visualization of density and the final mesh at the end of the simulation, for both the adaptation strategies which demonstrates the capability of the algorithm to perform dynamic adaptation with any element type separately and also on hybrid grids. It can be observed that the use of strategy 2 refinement methods the solutions are comparatively smoother than the refinement of grids with strategy1.

In figure 7 we have the density plots compared with increasing order of numerical schemes which helps us to study the performance of AMR with higher order schemes. These results are compared to t an analytical solution to check for the accuracy of the schemes with AMR. It can be observed that all the numerical schemes capture the peak with all the grids except for quads which tend to over predict with a weno5 numerical scheme on both the strategies. The quad grid with both the strategies and with higher order numerical schemes fail to capture the total diffusivity happening in the simulation, but they tend to stay smooth throughout. On the other hand with the hybrid and triangular grids the diffusivity capturing is better and closer to the reference solution but comes with a noise generated due to the weno weights of the triangular elements present in the grid. It is important to notice that the solution is less susceptible to noise with hybrid and tri grids with the use of the strategy 2 for refinement.

5.2 Forward Facing Step Test Case

The forward facing step is a classic test case to study the flow separation and recirculation caused by a sudden contraction in a channel. For this case depending on the ratio of the boundary layer thickness at the step to the step of the height there might be one to three recirculation regions, one upstream, the other in the downstream and the other just immediately above the step, the flow might also separate from the upper sharp corner of the step generating a recirculation region behind the step.

For the case under consideration a domain where length (L) of the domain is equal to 3 units and the height (H) of the domain is equal to 1 unit, the step is located at (l) 0.2 units from the inlet and with a height (hl) of 0.2 units is considered,

$$
\begin{aligned}
&0 \le x \le L, \begin{cases} y = 0 \\ y = H \end{cases} \{\phi \\
&0 \le y \le H, \begin{cases} x = 0 \\ x = L \end{cases}
\end{aligned} \tag{4}
$$

$where, \phi = u, v, p$

The above equation (4) describes the boundary conditions, where the boundaries of the geometry are considered to be a wall, friction is created at this wall due to the viscosity and momentum of the fluid (air). The inlet is taken as the velocity inlet where the velocity is given in the u direction equal to mach 3, the outlet of the channel is considered as the pressure outlet. The pressure is taken as 1atm and the density as 1.4 kg/m^3 for the problem which is solved using the Euler equations as mentioned in equations (1) and (2).

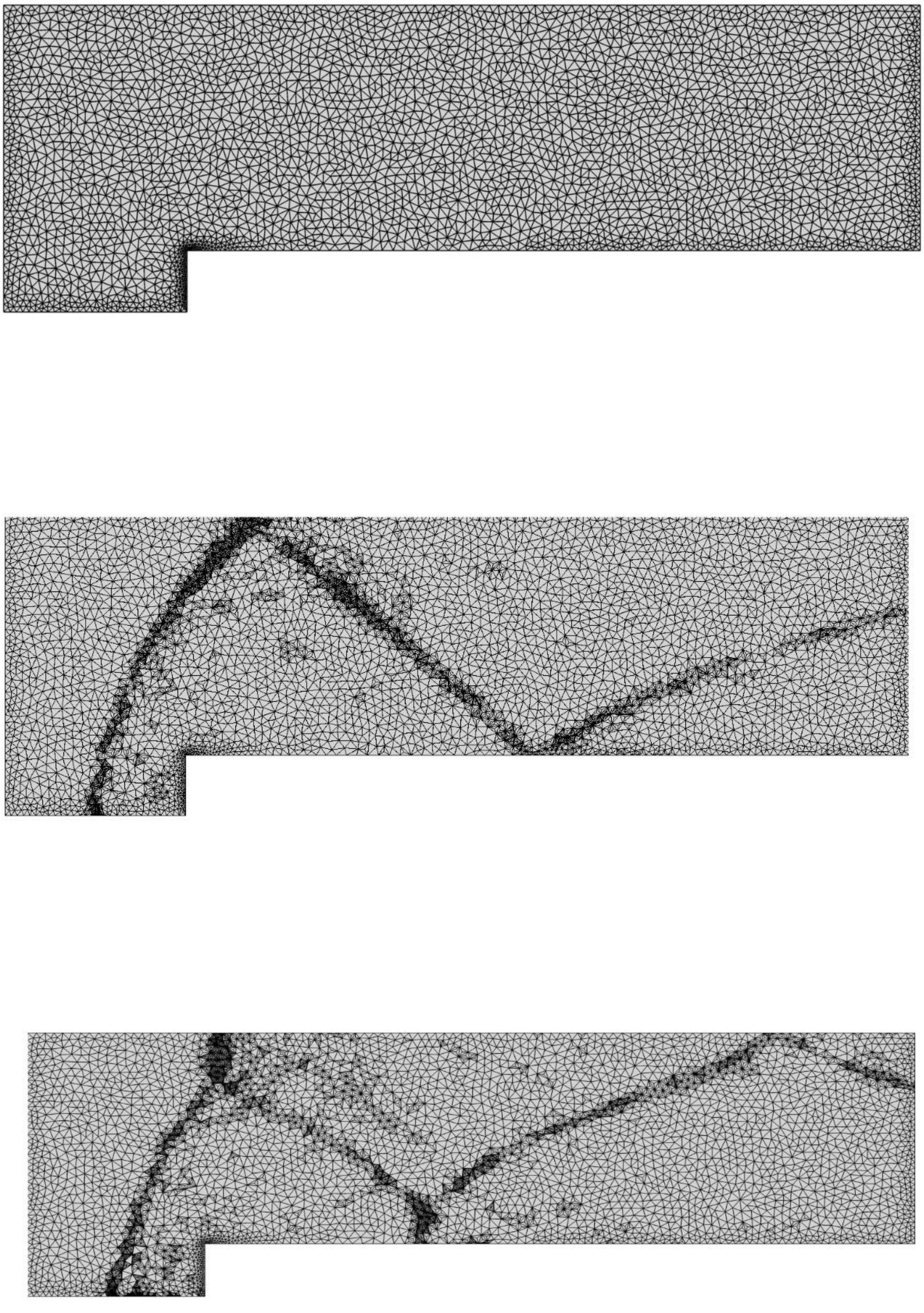


Figure 8. Forward facing step, initial mesh, mest at t=2.0secs, final mesh

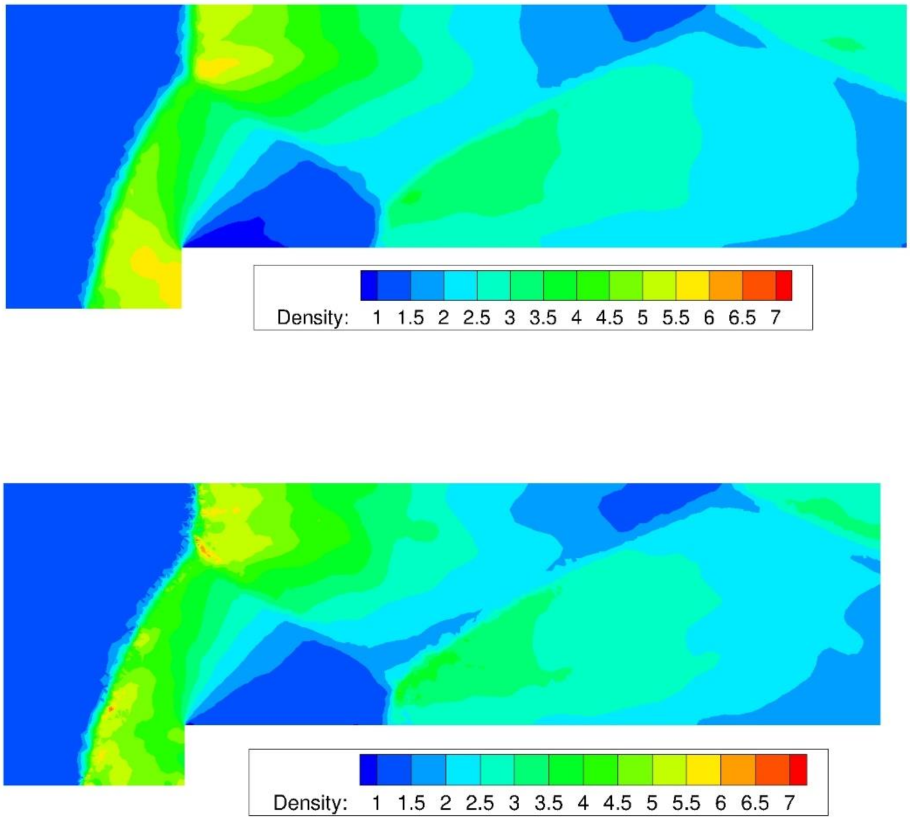


Figure 9. Contours of density, with no-refinement and with refinement

The domain is made up of triangular mesh which are moderately coarse for this setup. The simulation is run with a weno3 numerical scheme and a moderately aggressive AMR setup with second, third and fourth levels of refinement for a total simulation time of 4 seconds and AMR being activated every 0.9 seconds based on the gradients of density form the simulation, hence we would have five levels of adaptation happening in the course of the simulation. Figure 8 shows the triangular mesh of the case at the initial instance, mesh with adaptation at t=2.0secs and the mesh at t=4.0 secs.

When the flow enters through the left boundary at the velocity of Mach 3 and exits through the right pressure outlet boundary. During the first few time steps a strong shock is generated due to the step and an expansion wave appears originating from the top corner of the step. The shock then moves progressively towards the top wall and gets reflected at the wall, after a few tome steps the flow exhibits several shock wave reflections at the top and bottom walls before the shock exits the domain. Figure 9 shows the contours of density with a case run with no refinement and the case run with the AMR. It can be observed that the AMR helps in capturing the propagating shock with a better definition and also helps in preserving solution accuracy at places where the gradients are not that steep. This case clearly demonstrates the capability of tracking steep gradients by the AMR by accurately capturing the shock being propagated through the domain.

6 CONCLUSION

This paper proposed a mesh adaptation procedure for hybrid unstructured grids that gives a conformal mesh with node based refinement which follows a tree based hierarchical data structure where the decision on adaptation is always taken at the parent level. The quality of the refined mesh is kept comparatively high even with a few irregular elements present in the grid. From the test cases considered it can be said that the AMR efficiently performs the refinement where required and coarsens where it is not. The advantage of this method developed here is that the refinement or the coarsening process can be done as local as possible and also can be done per cell basis which gives the user more control over the process and is very essential and would aid in better parallelisation of the algorithm.

Solution transfer or remapping of solution of from child to parent and vice versa has always been a concern for conserving the integral and pointwise quantities of the solution, an efficient method has been devised with this algorithm where solution averaging based on volume weights have been done for remapping solution between the different levels of refinement and solution from the parent is remapped as a whole to the child cells, hence conserving the quantities. It was observed that the quadrilateral refinement was quite smooth than the triangle and hybrid refinement but failed to capture the diffusivity in the solution. Strategy 2 is a better option for refinement as there was less noise observed with triangular and hybrid grids.

This algorithm in conjunction with the high order solver can be used for solving a variety of problems pertaining to capturing of steep gradients and also can perform well with RANS cases. The present work will be extended to 3D domain for hexahedral, tetrahedral and prism elements and also the AMR algorithm would be parallelized.

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