ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

ENERGY AND MOMENTUM CONSERVING VARIATIONAL BASED TIME INTEGRATION OF ANISOTROPIC HYPERELASTIC CONTINUA

Michael Groß[1], Rajesh Ramesh[2] and Julian Dietzsch[3]

Technische Universität Chemnitz, Professorship of applied mechanics and dynamics
Reichenhainer Straße 70, D-09126 Chemnitz
[1] michael.gross@mb.tu-chemnitz.de, [2] rajesh.ramesh@mb.tu-chemnitz.de, [3] tmd@mb.tu-chemnitz.de

Keywords: Time stepping schemes, finite elasticity, anisotropy, fiber-reinforced continua.

Abstract. *For many years, the importance of fiber-reinforced polymers is steadily increasing in mechanical engineering. According to the high strength in fiber direction, these composites replace more and more traditional homogeneous materials, especially in lightweight structures. Fiber-reinforced material parts are often manufactured from carbon fibers as pure attachment parts, or from steel for transmitting forces. Whereas attachment parts are mostly subjected to small deformations, force transmission parts usually suffer large deformations in at least one direction. For the latter, a geometrically non-linear formulation of these anisotropic continua is indispensable [1]. A familar example is a rotor blade, in which the fibers possess the function of stabilizing the structure in order to counteract large centrifugal forces. For long-run numerical analyses of rotor blade motions, we have to apply numerically stable and robust time integration schemes for anisotropic continua.*

This paper is an extension of Reference [2], which is in turn an extension of Reference [3] to a special anisotropic material class, namely a transversely isotropic hyperelastic material based on the wellknown concept of structural tensors. In Reference [3], higher-order accurate time-stepping schemes are developed systematically with the focus on numerical stability and robustness in the presence of stiffness combined with large rotations for computing large motions. In the former work, these advantages over conventional time stepping schemes are combined with highly non-linear anisotropic material formulated with polyconvex free energy density functions [4]. The corresponding time integrators preserve all conservation laws of a free motion of a hyperelastic continuum, which means the total linear and the total angular momentum conservation law as well as the total energy conservation law. Both are numerically advantageous, because it guarantees that the discrete configuration vector is embedded in the physically consistent solution space. In order to guarantee the preservation of the total energy, the transient approximation of the anisotropic stress tensor is superimposed with an algorithmic stress field based on an assumed 'strain' field.

The presented numerical examples show the behaviour of the non-linear anisotropic material in Reference [4] under static and transient loads, their conservation laws and the higher-order accuracy of the variational based time approximation.

1 INTRODUCTION

We begin by summarizing the kinematical aspects of the considered transversely isotropic continuum. In Fig. 1 on the right-hand side, we show the reference configuration $\mathscr{B}_0$ of the considered fiber-reinforced continuum body $\mathscr{B}$. The configuration $\mathscr{B}_0 = \mathscr{B}_0^M \cup \mathscr{B}_0^F$ is defined

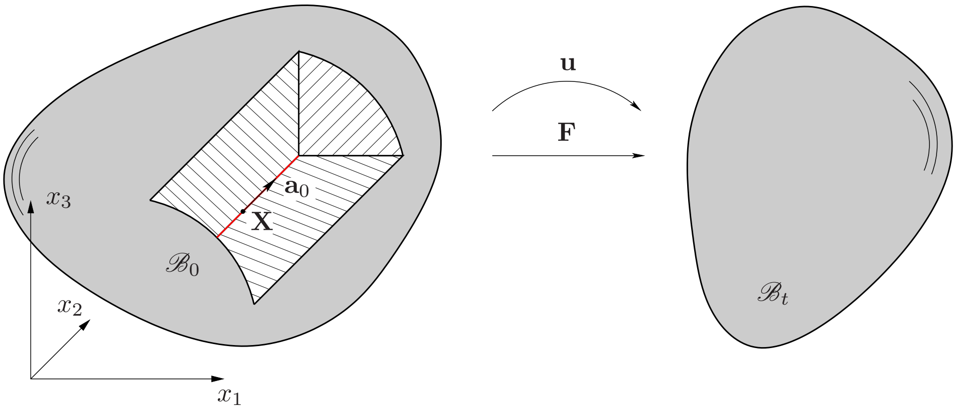


Figure 1: Reference and current configuration of a transversely isotropic continuum.

as the *homogenized* union of the set $\mathscr{B}_0^M$ for the matrix and the set $\mathscr{B}_0^F$ for the fibers. The imaginary fiber at *any* point $\mathbf{X} \in \mathscr{B}_0$ is directed along the *normalized vector* $\mathbf{a}_0$. Since we assume that both subsets are perfectly connected, the corresponding stretched vector $\mathbf{a}$ in the deformed configuration $\mathscr{B}_t$ is given by

$$\mathbf{a} = \mathbf{F}\,\mathbf{a}_0 \tag{1}$$

where

$$\mathbf{F} := \nabla \mathbf{u} + \mathbf{I} \tag{2}$$

denotes the deformation gradient of $\mathscr{B}_0$ and $\mathbf{u}$ the displacement vector field. The tensor $\mathbf{I}$ designates the second-order unit tensor. The symbol ∇ denotes the partial derivative with respect to the material point $\mathbf{X} \in \mathscr{B}_0$. The deformation gradient $\mathbf{F}_F$ of the fiber continuum $\mathscr{B}_0^F$ then takes the form

$$\mathbf{F}_F := \mathbf{a} \otimes \mathbf{a}_0 = \mathbf{F}\,\mathbf{a}_0 \otimes \mathbf{a}_0 = \mathbf{F}\,\mathbf{A}_0 \tag{3}$$

where

$$\mathbf{A}_0 := \mathbf{a}_0 \otimes \mathbf{a}_0 \tag{4}$$

designates the structural tensor of the fiber reference configuration $\mathscr{B}_0^F$. The corresponding right CAUCHY-GREEN tensor $\mathbf{C}_F$ then reads

$$\mathbf{C}_F := \mathbf{F}_F^T\,\mathbf{F}_F = [\mathbf{F}\,\mathbf{a}_0 \otimes \mathbf{a}_0]^T\,[\mathbf{F}\,\mathbf{a}_0 \otimes \mathbf{a}_0] = [\mathbf{a}_0 \otimes \mathbf{a}_0]\,\mathbf{C}\,[\mathbf{a}_0 \otimes \mathbf{a}_0] = \mathbf{A}_0\,\mathbf{C}\,\mathbf{A}_0 = [\mathbf{C} : \mathbf{A}_0]\,\mathbf{A}_0 \tag{5}$$

where $\mathbf{C} := \mathbf{F}^T\,\mathbf{F}$ denotes the right CAUCHY-GREEN tensor of $\mathscr{B}_0$. Based on these deformation measures, we consider the strain energy function W of the considered transversely isotropic elastic continuum on the one hand (i) as the *unpartitioned* function $W(\mathbf{C}; \mathbf{A}_0, \boldsymbol{\kappa}_0)$, where the semicolon in the argument separates the *parameter* $\mathbf{A}_0$ and $\boldsymbol{\kappa}_0$, *acting at any* $\mathbf{X} \in \mathscr{B}_0$, from the variable $\mathbf{C}$, and on the other hand (ii) as the *partitioned* function

$$W(\mathbf{C}; \mathbf{A}_0, \boldsymbol{\kappa}_0) = W_M(\mathbf{C}; \boldsymbol{\kappa}_{0_M}) + W_F(\mathbf{C}_F; \boldsymbol{\kappa}_{0_F}) \tag{6}$$

The parameter $\boldsymbol{\kappa}_0$, $\boldsymbol{\kappa}_{0_M}$ and $\boldsymbol{\kappa}_{0_F}$ are vectors including material constants with respect to $\mathscr{B}_0$. The second PIOLA-KIRCHHOFF stress tensor $\mathbf{S}$ corresponding to Eq. (6) is given by

$$
\begin{aligned}
\mathbf{S} \equiv 2\,\frac{\partial W(\mathbf{C};\mathbf{A}_0,\boldsymbol{\kappa}_0)}{\partial \mathbf{C}} &= \mathbf{S}_M + \mathbf{S}_F && (7)\\
&= 2\,\mathrm{D}W_M(\mathbf{C};\boldsymbol{\kappa}_{0_M}) + 2\,\mathrm{D}W_F(\mathbf{C}_F;\boldsymbol{\kappa}_{0_M}) : \frac{\partial \mathbf{C}_F}{\partial \mathbf{C}}\\
&= 2\,\mathrm{D}W_M(\mathbf{C};\boldsymbol{\kappa}_{0_M}) + 2\,\mathbf{A}_0\,\mathrm{D}W_F(\mathbf{C}_F;\boldsymbol{\kappa}_{0_M})\,\mathbf{A}_0 && (8)\\
\mathbf{S} &= 2\,\mathrm{D}W_M(\mathbf{C};\boldsymbol{\kappa}_{0_M}) + 2\,\left[\mathrm{D}W_F(\mathbf{C}_F;\boldsymbol{\kappa}_{0_M}) : \mathbf{A}_0\right]\mathbf{A}_0 && (9)
\end{aligned}
$$

The notation $\mathrm{D}(\bullet)$ denotes the FRÉCHET derivative of a volume density with respect to its argument. The strain energy functions W of $\mathscr{B}_0$, W_M of $\mathscr{B}_0^M$ or W_F of $\mathscr{B}_0^F$, respectively, directly depends on the *invariants* of the corresponding right CAUCHY-GREEN tensors. We assume (i) the unpartitioned case

$$
W(\mathbf{C};\mathbf{A}_0,\boldsymbol{\kappa}_0) = \hat{W}(I_1, I_2, I_3, I_4;\boldsymbol{\kappa}_0) \tag{10}
$$

and (ii) the partitioned case

$$
W_M = \hat{W}_M(I_1, I_2, I_3;\boldsymbol{\kappa}_{0_M}) \qquad\qquad W_F = \hat{W}_F(I_4;\boldsymbol{\kappa}_{0_F}) \tag{11}
$$

where

$$
I_1 := \mathbf{C} : \mathbf{I} \qquad\qquad I_2 := \tfrac{1}{2}\left[(I_1)^2 - \mathbf{C}^2 : \mathbf{I}\right] \qquad\qquad I_3 := \det \mathbf{C} \tag{12}
$$

denotes the tensor invariants of the right CAUCHY-GREEN tensors $\mathbf{C}$, and

$$
I_4 \equiv \mathbf{C}_F : \mathbf{A}_0 = \mathbf{A}_0\,\mathbf{C}\,\mathbf{A}_0 : \mathbf{A}_0 = [\mathbf{a}_0 \cdot \mathbf{C}\,\mathbf{a}_0]\,\mathbf{a}_0 \cdot \mathbf{a}_0 = \mathbf{a}_0 \cdot \mathbf{C}\,\mathbf{a}_0 = \mathbf{C} : \mathbf{A}_0 = \mathbf{a} \cdot \mathbf{a} =: C_F \tag{13}
$$

the squared fiber stretch $C_F \equiv \lambda_F^2$. Using the fourth invariant $I_4 =: C_F$, the right CAUCHY-GREEN tensor $\mathbf{C}_F$ and the second PIOLA-KIRCHHOFF stress tensor $\mathbf{S}_F$ of the partitioned strain energy function, respectively, can be simply written as

$$
\mathbf{C}_F = C_F\,\mathbf{A}_0 \qquad\qquad \mathbf{S}_F = 2\,\left[\mathrm{D}\hat{W}_F(C_F;\boldsymbol{\kappa}_{0_F})\,\frac{\partial C_F}{\partial \mathbf{C}_F} : \mathbf{A}_0\right]\mathbf{A}_0 = 2\,\mathrm{D}\hat{W}_F(C_F;\boldsymbol{\kappa}_{0_F})\,\mathbf{A}_0 \tag{14}
$$

Hence, the directions of the fiber deformation tensor $\mathbf{C}_F$ and fiber stress tensor $\mathbf{S}_F$ are uniquely prescribed by the structure tensor $\mathbf{A}_0$, as expected.

2 EULER-LAGRANGE EQUATIONS

With regard to the numerical time integration, we now introduce *variationally consistent*

- temporally continuous assumed 'strains' $\tilde{\mathbf{C}}$ and $\tilde{C}_F$, as well as
- temporally discontinuous superimposed stresses $\tilde{\mathbf{S}}$ and $\tilde{S}_F$, respectively.

The former are necessary for an exact analytical time integration of *approximated* strain energy functions [6], and the later for their exact numerical time integration [8]. Hence, the superimposed stresses $\tilde{\mathbf{S}}$ and $\tilde{S}_F$ are responsible for the energy consistency of the discrete EULER-LAGRANGE equations, but they have to vanish identically for guaranteeing energy consistency of the continuous EULER-LAGRANGE equations. We consider the strain energy function

1. $W(\mathbf{C};\mathbf{A}_0,\boldsymbol{\kappa}_0)$ on $\mathscr{B}_0$ (unpartitioned strain energy), or
2. $W_M(\mathbf{C};\boldsymbol{\kappa}_{0_M})$ and $W_F(\mathbf{C}_F;\boldsymbol{\kappa}_{0_F})$ separately (partitioned strain energy).

We derive the continuous equations of motion by using a mixed *principle of virtual power* or *principle of Jourdain*, respectively, a differential variational principle [9]. The modivation for applying this principle from the outset is to satisfy the *total energy balance* in both the continuous as well as the discrete setting. Denoting by a superimposed dot the partial time derivative, in the unpartitioned case, this balance takes the form

$$\dot{\mathcal{T}}(\dot{\mathbf{u}},\dot{\mathbf{v}},\dot{\mathbf{p}};\rho_0)+\dot{\Pi}^{\mathrm{int}}(\dot{\mathbf{u}},\dot{\tilde{\mathbf{C}}},\mathbf{S};\mathbf{A}_0,\boldsymbol{\kappa}_0,\tilde{\mathbf{S}})=\int_{\mathscr{B}_0}\rho_0\mathbf{b}\cdot\dot{\mathbf{u}}\,\mathrm{d}V+\int_{\partial_t\mathscr{B}_0}\mathbf{t}\cdot\dot{\mathbf{u}}\,\mathrm{d}A+\int_{\partial_u\mathscr{B}_0}\mathbf{h}\cdot\left(\dot{\mathbf{u}}-\dot{\bar{\mathbf{u}}}\right)\mathrm{d}A \tag{15}$$

with the kinetic power

$$\dot{\mathcal{T}}(\dot{\mathbf{u}},\dot{\mathbf{v}},\dot{\mathbf{p}};\rho_0):=\int_{\mathscr{B}_0}[\rho_0\mathbf{v}-\mathbf{p}]\cdot\dot{\mathbf{v}}\,\mathrm{d}V-\int_{\mathscr{B}_0}\dot{\mathbf{p}}\cdot[\mathbf{v}-\dot{\mathbf{u}}]\,\mathrm{d}V+\int_{\mathscr{B}_0}\mathbf{p}\cdot\ddot{\mathbf{u}}\,\mathrm{d}V \tag{16}$$

where

$$\mathbf{p}=\rho_0\mathbf{v}\qquad\text{and}\qquad\mathbf{v}=\dot{\mathbf{u}} \tag{17}$$

denotes the linear momentum vector and the material velocity vector, respectively. The scalar ρ_0 denotes the mass density field in $\mathscr{B}_0$. The stress power $\dot{\Pi}^{\mathrm{int}}$ is written in dependence on the second PIOLA-KIRCHHOFF stress tensor $\mathbf{S}$, the superimposed stress tensor $\tilde{\mathbf{S}}$ and the assumed 'strain' tensor $\tilde{\mathbf{C}}$ as

$$\begin{aligned}\dot{\Pi}^{\mathrm{int}}(\dot{\mathbf{u}},\dot{\tilde{\mathbf{C}}},\mathbf{S};\mathbf{A}_0,\boldsymbol{\kappa}_0,\tilde{\mathbf{S}}):=\;&\frac{1}{2}\int_{\mathscr{B}_0}\left[2\,\mathrm{D}W(\tilde{\mathbf{C}};\mathbf{A}_0,\boldsymbol{\kappa}_0)+\tilde{\mathbf{S}}-\mathbf{S}\right]:\dot{\tilde{\mathbf{C}}}\,\mathrm{d}V\\&-\frac{1}{2}\int_{\mathscr{B}_0}\dot{\mathbf{S}}:\left[\tilde{\mathbf{C}}-\mathbf{C}(\mathbf{u})\right]\mathrm{d}V+\frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}:\dot{\mathbf{C}}(\dot{\mathbf{u}})\,\mathrm{d}V\\&=\int_{\mathscr{B}_0}\dot{W}\,\mathrm{d}V\end{aligned} \tag{18}$$

where

$$\mathbf{S}=2\,\mathrm{D}W(\tilde{\mathbf{C}};\mathbf{A}_0,\boldsymbol{\kappa}_0)+\tilde{\mathbf{S}}\qquad\qquad\tilde{\mathbf{C}}=\mathbf{C}(\mathbf{u}):=(\nabla\mathbf{u}+\mathbf{I})^T(\nabla\mathbf{u}+\mathbf{I}) \tag{19}$$

The superimposed stress tensor $\tilde{\mathbf{S}}=\mathbf{O}$, with the zero tensor $\mathbf{O}$, has to vanish for energy consistency. On the right-hand side of Eq. (15), there is the external power depending on the body force vector $\mathbf{b}$ per unit mass on $\mathscr{B}_0$, the traction force vector $\mathbf{t}$ per unit area on the NEUMANN boundary $\partial_t\mathscr{B}_0$ and the LAGRANGE multiplier vector $\mathbf{h}$ enforcing the constraint

$$\mathbf{u}-\bar{\mathbf{u}}=\mathbf{0}\quad\text{on}\quad\partial_u\mathscr{B}_0 \tag{20}$$

of a prescribed displacement $\bar{\mathbf{u}}$ on the DIRICHLET boundary $\partial_u\mathscr{B}_0$. Both boundary sets satisfy the conditions $\partial\mathscr{B}_0=\partial_t\mathscr{B}_0\cup\partial_u\mathscr{B}_0$ and $\partial_t\mathscr{B}_0\cap\partial_u\mathscr{B}_0=\emptyset$, where $\partial\mathscr{B}_0$ designates the boundary of the reference configuration.

2.1 The unpartitioned strain energy function

In the unpartitioned case, we introduce the assumed 'strain' field $\tilde{\mathbf{C}}$ and the superimposed stress field $\tilde{\mathbf{S}}$ for the entire reference configuration variationally consistent by considering the virtual power principle

$$\begin{aligned}&\delta_*\dot{\mathcal{H}}(\dot{\mathbf{u}},\dot{\mathbf{v}},\dot{\mathbf{p}},\dot{\tilde{\mathbf{C}}},\mathbf{S};\rho_0,\mathbf{A}_0,\boldsymbol{\kappa}_0,\mathbf{b},\mathbf{t},\dot{\bar{\mathbf{u}}},\tilde{\mathbf{S}}):=\\&\delta_*\dot{\mathcal{T}}(\dot{\mathbf{u}},\dot{\mathbf{v}},\dot{\mathbf{p}};\rho_0)+\delta_*\dot{\Pi}^{\text{int}}(\dot{\mathbf{u}},\dot{\tilde{\mathbf{C}}},\mathbf{S};\mathbf{A}_0,\boldsymbol{\kappa}_0,\tilde{\mathbf{S}})+\delta_*\dot{\Pi}^{\text{ext}}(\dot{\mathbf{u}};\rho_0,\mathbf{b},\mathbf{t},\dot{\bar{\mathbf{u}}})=0\end{aligned}\tag{21}$$

with the virtual kinetic power

$$\delta_*\dot{\mathcal{T}}(\dot{\mathbf{u}},\dot{\mathbf{v}},\dot{\mathbf{p}};\rho_0):=\int_{\mathscr{B}_0}[\rho_0\mathbf{v}-\mathbf{p}]\cdot\delta_*\dot{\mathbf{v}}\,\mathrm{d}V-\int_{\mathscr{B}_0}\delta_*\dot{\mathbf{p}}\cdot[\mathbf{v}-\dot{\mathbf{u}}]\,\mathrm{d}V+\int_{\mathscr{B}_0}\dot{\mathbf{p}}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}V\tag{22}$$

the virtual external power

$$\delta_*\dot{\Pi}^{\text{ext}}(\dot{\mathbf{u}};\rho_0,\mathbf{b},\mathbf{t},\dot{\bar{\mathbf{u}}}):=-\int_{\mathscr{B}_0}\rho_0\,\mathbf{b}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}V-\int_{\partial_t\mathscr{B}_0}\mathbf{t}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A-\int_{\partial_u\mathscr{B}_0}\mathbf{h}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A\tag{23}$$

and the virtual internal power

$$\begin{aligned}\delta_*\dot{\Pi}^{\text{int}}(\dot{\mathbf{u}},\dot{\tilde{\mathbf{C}}},\mathbf{S};\mathbf{A}_0,\boldsymbol{\kappa}_0,\tilde{\mathbf{S}}):=\;&\frac{1}{2}\int_{\mathscr{B}_0}\Big[2\mathrm{D}W(\tilde{\mathbf{C}};\mathbf{A}_0,\boldsymbol{\kappa}_0)+\tilde{\mathbf{S}}-\mathbf{S}\Big]:\delta_*\dot{\tilde{\mathbf{C}}}\,\mathrm{d}V\\&-\frac{1}{2}\int_{\mathscr{B}_0}\delta_*\mathbf{S}:\Big[\dot{\tilde{\mathbf{C}}}-\dot{\mathbf{C}}(\dot{\mathbf{u}})\Big]\,\mathrm{d}V+\frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}:\delta_*\dot{\mathbf{C}}(\dot{\mathbf{u}})\,\mathrm{d}V\end{aligned}\tag{24}$$

The symbol δ_* denotes the variation with respect to the variables (not the parameter behind the semicolon) in the function argument. Integration by parts in the last term of Eq. (24) furnishes

$$\frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}:\delta_*\dot{\mathbf{C}}(\dot{\mathbf{u}})\,\mathrm{d}V=\int_{\mathscr{B}_0}\mathbf{FS}:\nabla(\delta_*\dot{\mathbf{u}})\,\mathrm{d}V=\int_{\partial_t\mathscr{B}_0}\mathbf{FSN}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A-\int_{\mathscr{B}_0}\mathrm{D{\small IV}}[\mathbf{FS}]\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}V\tag{25}$$

The vector $\mathbf{N}$ denotes the normal field on the NEUMANN boundary $\partial_t\mathscr{B}_0$, and $\mathrm{D{\small IV}}[\bullet]$ the divergence operator with respect to $\mathbf{X}\in\mathscr{B}_0$. Rearranging termes in Eq. (21) according to the variations $\delta_*\dot{\mathbf{p}}$, $\delta_*\dot{\mathbf{v}}$, $\delta_*\mathbf{S}$, $\delta_*\dot{\tilde{\mathbf{C}}}$ and $\delta_*\dot{\mathbf{u}}$, we obtain the variational form

$$\begin{aligned}0\;=\;&\int_{\mathscr{B}_0}[\rho_0\mathbf{v}-\mathbf{p}]\cdot\delta_*\dot{\mathbf{v}}\,\mathrm{d}V-\int_{\mathscr{B}_0}\delta_*\dot{\mathbf{p}}\cdot[\mathbf{v}-\dot{\mathbf{u}}]\,\mathrm{d}V-\int_{\mathscr{B}_0}[\mathrm{D{\small IV}}[\mathbf{FS}]+\rho_0\mathbf{b}-\dot{\mathbf{p}}]\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}V\\&-\frac{1}{2}\int_{\mathscr{B}_0}\Big[\mathbf{S}-2\,\mathrm{D}W(\tilde{\mathbf{C}};\mathbf{A}_0,\boldsymbol{\kappa}_0)-\tilde{\mathbf{S}}\Big]:\delta_*\dot{\tilde{\mathbf{C}}}\,\mathrm{d}V-\frac{1}{2}\int_{\mathscr{B}_0}\delta_*\mathbf{S}:\Big[\dot{\tilde{\mathbf{C}}}-\dot{\mathbf{C}}(\dot{\mathbf{u}})\Big]\,\mathrm{d}V\\&-\int_{\partial_t\mathscr{B}_0}[\mathbf{t}-\mathbf{FSN}]\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A-\int_{\partial_u\mathscr{B}_0}\mathbf{h}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A\end{aligned}\tag{26}$$

Owing to the fundamental theorem of variational calculus, the corresponding continuous EULER-LAGRANGE equations read

$$\mathbf{v}=\dot{\mathbf{u}}\quad\text{with}\quad\mathbf{u}(t_0)=\mathbf{u}_0\tag{27}$$

$$\rho_0\mathbf{v}=\mathbf{p}\qquad\forall t\geqslant t_0\tag{28}$$

$$\mathrm{D{\small IV}}[\mathbf{FS}]+\rho_0\mathbf{b}=\dot{\mathbf{p}}\quad\text{with}\quad\mathbf{p}(t_0)=\mathbf{p}_0\equiv\rho_0\mathbf{v}_0\tag{29}$$

$$\dot{\mathbf{C}}(\dot{\mathbf{u}})=\dot{\tilde{\mathbf{C}}}\quad\text{with}\quad\tilde{\mathbf{C}}(t_0)=\mathbf{C}(\mathbf{u}_0)\equiv(\nabla\mathbf{u}_0+\mathbf{I})^T(\nabla\mathbf{u}_0+\mathbf{I})\tag{30}$$

$$2\,\mathrm{D}W(\tilde{\mathbf{C}};\mathbf{A}_0,\boldsymbol{\kappa}_0)+\tilde{\mathbf{S}}=\mathbf{S}\qquad\forall t\geqslant t_0\tag{31}$$

with the corresponding inital conditions in $\mathscr{B}_0$ as well as the boundary conditions

$$\mathbf{FSN} = \mathbf{t} \qquad \forall t \geqslant t_0 \quad \text{on } \partial_t\mathscr{B}_0 \tag{32}$$

$$\delta_*\dot{\mathbf{u}} = \mathbf{0} \Longleftrightarrow \dot{\mathbf{u}} = \dot{\bar{\mathbf{u}}} \text{ with } \mathbf{u}(t_0) = \bar{\mathbf{u}}(t_0) \quad \text{on } \partial_u\mathscr{B}_0 \tag{33}$$

Consequently, the PIOLA-KIRCHHOFF stress tensor field $\mathbf{S}$ is temporally discontinuous, but the displacement vector field $\mathbf{u}$, the material velocity field $\mathbf{v}$, the linear momentum field $\mathbf{p}$ *as well as* the assumed 'strain' field $\tilde{\mathbf{C}}$ are temporally continuous. The superimposed stress tensor $\tilde{\mathbf{S}}$ has to vanish in these EULER-LAGRANGE equations for satisfying the total energy balance in Eq. (15).

2.2 The partitioned strain energy function

In the partitioned formulation, we introduce the assumed 'strain' tensor $\tilde{\mathbf{C}}$ and the superimposed stress tensor $\tilde{\mathbf{S}}_M$ on $\mathscr{B}_0^M$ and the assumed 'strain' $\tilde{C}_F$ and the superimposed fiber stress $\tilde{S}_F$ on $\mathscr{B}_0^F$ by considering the virtual power principle

$$\begin{aligned}&\delta_*\dot{\mathcal{H}}(\dot{\mathbf{u}}, \dot{\mathbf{v}}, \dot{\mathbf{p}}, \dot{\tilde{\mathbf{C}}}, \mathbf{S}_M, \dot{\tilde{C}}_F, \mathbf{S}_F; \rho_0, \boldsymbol{\kappa}_{0_M}, \boldsymbol{\kappa}_{0_F}, \mathbf{A}_0, \mathbf{b}, \mathbf{t}, \dot{\bar{\mathbf{u}}}, \tilde{\mathbf{S}}_M, \tilde{S}_F) := \\ &\delta_*\dot{\mathcal{T}}(\dot{\mathbf{u}}, \dot{\mathbf{v}}, \dot{\mathbf{p}}; \rho_0) + \delta_*\dot{\Pi}^{\text{ext}}(\dot{\mathbf{u}}; \rho_0, \mathbf{b}, \mathbf{t}, \dot{\bar{\mathbf{u}}}) + \delta_*\dot{\Pi}^{\text{int}}(\dot{\mathbf{u}}, \dot{\tilde{\mathbf{C}}}, \mathbf{S}_M, \dot{\tilde{C}}_F, \mathbf{S}_F; \boldsymbol{\kappa}_{0_M}, \boldsymbol{\kappa}_{0_F}, \mathbf{A}_0, \tilde{\mathbf{S}}_M, \tilde{S}_F) = 0\end{aligned} \tag{34}$$

The virtual kinetic power is identical to Eq. (22) and the virtual external power is identical to Eq. (23). But the virtual internal power is now given by

$$\begin{aligned}&\delta_*\dot{\Pi}^{\text{int}}(\dot{\mathbf{u}}, \dot{\tilde{\mathbf{C}}}, \mathbf{S}_M, \dot{\tilde{C}}_F, \mathbf{S}_F; \boldsymbol{\kappa}_{0_M}, \boldsymbol{\kappa}_{0_F}, \mathbf{A}_0, \tilde{\mathbf{S}}_M, \tilde{S}_F) := \\ &\frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\mathrm{D}W_M(\tilde{\mathbf{C}}; \boldsymbol{\kappa}_{0_M}) + \tilde{\mathbf{S}}_M - \mathbf{S}_M\Big] : \delta_*\dot{\tilde{\mathbf{C}}}\,\mathrm{d}V \\ &\frac{1}{2}\int_{\mathscr{B}_0}\Big[2\,\mathrm{D}\hat{W}_F(\tilde{C}_F; \boldsymbol{\kappa}_{0_F}) + \tilde{S}_F - \mathbf{S}_F : \mathbf{A}_0\Big]\,\delta_*\dot{\tilde{C}}_F\,\mathrm{d}V \\ &-\frac{1}{2}\int_{\mathscr{B}_0}\delta_*\mathbf{S}_M : \Big[\dot{\tilde{\mathbf{C}}} - \dot{\mathbf{C}}(\dot{\mathbf{u}})\Big]\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}_M : \delta_*\dot{\mathbf{C}}(\dot{\mathbf{u}})\,\mathrm{d}V \\ &-\frac{1}{2}\int_{\mathscr{B}_0}\delta_*\mathbf{S}_F : \Big[\dot{\tilde{C}}_F\,\mathbf{A}_0 - \dot{\mathbf{C}}_F(\dot{\mathbf{u}})\Big]\,\mathrm{d}V + \frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}_F : \delta_*\dot{\mathbf{C}}_F(\dot{\mathbf{u}})\,\mathrm{d}V\end{aligned} \tag{35}$$

with

$$\mathbf{C}_F(\mathbf{u}) := [(\nabla\mathbf{u} + \mathbf{I})\,\mathbf{A}_0]^T\,[\nabla\mathbf{u} + \mathbf{I}]\,\mathbf{A}_0 = \mathbf{A}_0(\nabla\mathbf{u} + \mathbf{I})^T(\nabla\mathbf{u} + \mathbf{I})\,\mathbf{A}_0 \tag{36}$$

Bearing in mind the identity

$$\frac{1}{2}\mathbf{S}_F : \delta_*\dot{\mathbf{C}}_F(\dot{\mathbf{u}}) = \frac{1}{2}\mathbf{S}_F : \mathbf{A}_0\,\delta_*\dot{\mathbf{C}}(\dot{\mathbf{u}})\,\mathbf{A}_0 = \frac{1}{2}\mathbf{F}\,\mathbf{A}_0\,\mathbf{S}_F\,\mathbf{A}_0 : \nabla\dot{\mathbf{u}} = \mathbf{F}_F\,[\mathbf{S}_F : \mathbf{A}_0] : \nabla\dot{\mathbf{u}} \tag{37}$$

integration by parts leads to

$$\frac{1}{2}\int_{\mathscr{B}_0}\mathbf{S}_F : \delta_*\dot{\mathbf{C}}_F(\dot{\mathbf{u}})\,\mathrm{d}V = \int_{\partial_t\mathscr{B}_0}\mathbf{F}_F\,[\mathbf{S}_F : \mathbf{A}_0]\,\mathbf{N}\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}A - \int_{\mathscr{B}_0}\mathrm{DIV}\,[\mathbf{F}_F\,(\mathbf{S}_F : \mathbf{A}_0)]\cdot\delta_*\dot{\mathbf{u}}\,\mathrm{d}V \tag{38}$$

Again, by rearranging the terms in Eq. (34) due to the independent variations $\delta_*\dot{\mathbf{p}}$, $\delta_*\dot{\mathbf{v}}$, $\delta_*\mathbf{S}_M$, $\delta_*\dot{\tilde{\mathbf{C}}}$, $\delta_*\mathbf{S}_F$, $\delta_*\dot{\tilde{C}}_F$ and $\delta_*\dot{\mathbf{u}}$, we obtain the variational form

$$
\begin{aligned}
0 = \int_{\mathscr{B}_0} [\rho_0 \mathbf{v} - \mathbf{p}] \cdot \delta_* \dot{\mathbf{v}} \, \mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \dot{\mathbf{p}} \cdot [\mathbf{v} - \dot{\mathbf{u}}] \, \mathrm{d}V \\
- \int_{\mathscr{B}_0} [\mathrm{DIV}[\mathbf{F}\mathbf{S}_M + \mathbf{F}_F\,(\mathbf{S}_F : \mathbf{A}_0)] + \rho_0 \mathbf{b} - \dot{\mathbf{p}}] \cdot \delta_* \dot{\mathbf{u}} \, \mathrm{d}V \\
- \int_{\partial_t \mathscr{B}_0} [\mathbf{t} - (\mathbf{F}\mathbf{S}_M + \mathbf{F}_F\,(\mathbf{S}_F : \mathbf{A}_0))\,\mathbf{N}] \cdot \delta_* \dot{\mathbf{u}} \, \mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \mathbf{h} \cdot \delta_* \dot{\mathbf{u}} \, \mathrm{d}A \\
- \frac{1}{2} \int_{\mathscr{B}_0} \Big[\mathbf{S}_M - 2\,\mathrm{D}W_M(\tilde{\mathbf{C}}; \boldsymbol{\kappa}_{0_M}) - \tilde{\mathbf{S}}_M \Big] : \delta_* \dot{\tilde{\mathbf{C}}} \, \mathrm{d}V - \frac{1}{2} \int_{\mathscr{B}_0} \Big[\dot{\tilde{\mathbf{C}}} - \dot{\mathbf{C}}(\dot{\mathbf{u}}) \Big] : \delta_* \mathbf{S}_M \, \mathrm{d}V \\
- \int_{\mathscr{B}_0} \Big[\mathbf{S}_F : \mathbf{A}_0 - 2\,\mathrm{D}\hat{W}_F(\tilde{C}_F; \boldsymbol{\kappa}_{0_F}) - \tilde{S}_F \Big] \frac{\delta_* \dot{\tilde{C}}_F}{2} \, \mathrm{d}V - \frac{1}{2} \int_{\mathscr{B}_0} \Big[\dot{\tilde{C}}_F \, \mathbf{A}_0 - \dot{\mathbf{C}}_F(\dot{\mathbf{u}}) \Big] : \delta_* \mathbf{S}_F \, \mathrm{d}V
\end{aligned}
\tag{39}
$$

Taking the fundamental theorem of variational calculus into account, we arrive at the EULER-LAGRANGE equations

$$
\begin{aligned}
\mathbf{v} = \; & \dot{\mathbf{u}} && \text{with} \quad \mathbf{u}(t_0) = \mathbf{u}_0 && (40) \\
\rho_0 \mathbf{v} = \; & \mathbf{p} && \forall t \geqslant t_0 && (41) \\
\mathrm{DIV}\,[\mathbf{F}\mathbf{S}_M + \mathbf{F}_F\,(\mathbf{S}_F : \mathbf{A}_0)] + \rho_0 \mathbf{b} = \; & \dot{\mathbf{p}} && \text{with} \quad \mathbf{p}(t_0) = \mathbf{p}_0 \equiv \rho_0 \mathbf{v}_0 && (42) \\
\dot{\mathbf{C}}(\dot{\mathbf{u}}) = \; & \dot{\tilde{\mathbf{C}}} && \text{with} \quad \tilde{\mathbf{C}}(t_0) = \mathbf{C}(\mathbf{u}_0) && (43) \\
\dot{\mathbf{C}}_F(\dot{\mathbf{u}}) : \mathbf{A}_0 = \; & \dot{\tilde{C}}_F && \text{with} \quad \tilde{C}_F(t_0) = \mathbf{C}_F(\mathbf{u}_0) : \mathbf{A}_0 && (44) \\
2\,\mathrm{D}W_M(\tilde{\mathbf{C}}; \boldsymbol{\kappa}_{0_M}) + \tilde{\mathbf{S}}_M = \; & \mathbf{S}_M && \forall t \geqslant t_0 && (45) \\
\Big[2\,\mathrm{D}\hat{W}_F(\tilde{C}_F; \boldsymbol{\kappa}_{0_F}) + \tilde{S}_F \Big] \mathbf{A}_0 = \; & \mathbf{S}_F && \forall t \geqslant t_0 && (46)
\end{aligned}
$$

with $\tilde{\mathbf{S}}_M := \mathbf{O}$ and $\tilde{S}_F := 0$ for satisfying the total energy balance, and the boundary conditions

$$
\begin{aligned}
[\mathbf{F}\mathbf{S}_M + \mathbf{F}_F\,(\mathbf{S}_F : \mathbf{A}_0)]\,\mathbf{N} = \; & \mathbf{t} && \forall t \geqslant t_0 \quad \text{on} \;\; \partial_t \mathscr{B}_0 && (47) \\
\delta_* \dot{\mathbf{u}} = \mathbf{0} \Longleftrightarrow \dot{\mathbf{u}} = \; & \dot{\bar{\mathbf{u}}} \;\; \text{with} \;\; \mathbf{u}(t_0) = \bar{\mathbf{u}}(t_0) && \text{on} \;\; \partial_u \mathscr{B}_0 && (48)
\end{aligned}
$$

Consequently, the PIOLA-KIRCHHOFF stress fields $\mathbf{S}_M$ and $\mathbf{S}_F$ are temporally discontinuous, but the assumed 'strain' fields $\tilde{\mathbf{C}}$ and $\tilde{C}_F$ are again temporally continuous.

3 FULLY-DISCRETE WEAK FORMULATION

Next, we derive the temporally and spatially discrete weak variational formulation. In this section, we restrict ourselves to a linear piecewise continuous time approximation in $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$ (compare Reference [5]) as well as $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{C}}_F$, in order to demonstrate the consistency of the variational derivation of the assumed 'strain' approximations and the superimposed stress tensors $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{S}}_F$ with Reference [8]. But note that in the unpartitioned case both approximations are new for *higher-order time approximations*.

3.1 The unpartitioned strain energy function

The time integrator presented here follows from considering *the virtual power principle at collocation points* ξ_i of the time interval $[t_0, t_N]$ of interest, introduced by a time integration

$$\int_{t_0}^{t_N} \delta_* \dot{\mathcal{H}}(\dot{\mathbf{u}}(t), \dot{\mathbf{v}}(t), \dot{\mathbf{p}}(t), \dot{\tilde{\mathbf{C}}}(t), \mathbf{S}(t); \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}(t), \mathbf{t}(t), \dot{\bar{\mathbf{u}}}(t), \tilde{\mathbf{S}}(t)) \, \mathrm{d}t =$$
$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \delta_* \dot{\mathcal{H}}(\dot{\mathbf{u}}^n(t), \dot{\mathbf{v}}^n(t), \dot{\mathbf{p}}^n(t), \dot{\tilde{\mathbf{C}}}^n(t), \mathbf{S}^n(t); \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}^n(t), \mathbf{t}^n(t), \dot{\bar{\mathbf{u}}}^n(t), \tilde{\mathbf{S}}^n(t)) \, \mathrm{d}t \approx$$
$$\sum_{n=0}^{N-1} \int_{0}^{1} \delta_* \dot{\mathcal{H}}(\dot{\mathbf{u}}_h^n(\alpha), \dot{\mathbf{v}}_h^n(\alpha), \dot{\mathbf{p}}_h^n(\alpha), \dot{\tilde{\mathbf{C}}}_h^n(\alpha), \mathbf{S}_h^n(\alpha); \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}_h^n(\alpha), \mathbf{t}_h^n(\alpha), \dot{\bar{\mathbf{u}}}_h^n(\alpha), \tilde{\mathbf{S}}_h^n(\alpha)) h_n \mathrm{d}\alpha \approx$$
$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}(\dot{\mathbf{u}}_h^n(\xi_1), \dot{\mathbf{v}}_h^n(\xi_1), \dot{\mathbf{p}}_h^n(\xi_1), \dot{\tilde{\mathbf{C}}}_h^n(\xi_1), \mathbf{S}_h^n(\xi_1); \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}_h^n(\xi_1), \mathbf{t}_h^n(\xi_1), \dot{\bar{\mathbf{u}}}_h^n(\xi_1), \tilde{\mathbf{S}}_h^n(\xi_1)) \, h_n$$
$$\sum_{n=0}^{N-1} \delta_* \dot{\mathcal{H}}_d(\mathbf{u}_{n+1}, \mathbf{v}_{n+1}, \mathbf{p}_{n+1}, \tilde{\mathbf{C}}_{n+1}, \mathbf{S}_{n+\frac{1}{2}}; \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}_{n+\frac{1}{2}}, \mathbf{t}_{n+\frac{1}{2}}, \bar{\mathbf{u}}_{n+1}, \tilde{\mathbf{S}}_{n+\frac{1}{2}}) \, h_n \doteq 0 \qquad (49)$$

and the normalized time $\alpha \in [0, 1]$ via the linear transformation

$$\tau : [t_n, t_{n+1}] \ni t \mapsto t_n + \alpha \, (t_{n+1} - t_n) = t_n + \alpha \, h_n \qquad (50)$$

with respect to the time step size h_n, and after applying the midpoint rule with the one Gauss point $\xi_1 = \frac{1}{2}$ to the mentioned piecewise linear time approximations

$$\mathbf{u}_h^n(\alpha) := \mathbf{u}_n + \alpha \, (\mathbf{u}_{n+1} - \mathbf{u}_n) \qquad \mathbf{v}_h^n(\alpha) := \mathbf{v}_n + \alpha \, (\mathbf{v}_{n+1} - \mathbf{v}_n) \qquad (51)$$
$$\mathbf{p}_h^n(\alpha) := \mathbf{p}_n + \alpha \left(\mathbf{p}_{n+1} - \mathbf{p}_n\right) \qquad \tilde{\mathbf{C}}_h^n(\alpha) := \mathbf{C}_n + \alpha \, (\mathbf{C}_{n+1} - \mathbf{C}_n) \qquad (52)$$

In the following, we use the common finite difference notation $(\bullet)_{n+\frac{1}{2}}$ for symbols $(\bullet)_h^n(\frac{1}{2})$. Without integrating by parts but rearranging termes in Eq. (49) according to the independent variations, we obtain the semi-discrete variational form

$$\begin{aligned} 0 \; = \; & \int_{\mathscr{B}_0} \left[\rho_0 \mathbf{v}_{n+\frac{1}{2}} - \mathbf{p}_{n+\frac{1}{2}}\right] \cdot \delta_* \mathbf{v}_{n+1} \, \mathrm{d}V - \int_{\mathscr{B}_0} \delta_* \mathbf{p}_{n+1} \cdot \left[\mathbf{v}_{n+\frac{1}{2}} - \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h_n}\right] \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \left[\mathbf{S}_{n+\frac{1}{2}} - 2 \, \mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) - \tilde{\mathbf{S}}_{n+\frac{1}{2}}\right] : \delta_* \mathbf{C}_{n+1} \, \mathrm{d}V \\ & - \frac{1}{2} \int_{\mathscr{B}_0} \left[\mathbf{C}_{n+1} - \mathbf{C}_n - (\mathbf{F}_{n+1} + \mathbf{F}_n)^T (\mathbf{F}_{n+1} - \mathbf{F}_n)\right] : \delta_* \mathbf{S}_{n+\frac{1}{2}} \, \mathrm{d}V \\ & + \int_{\mathscr{B}_0} \left[\frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{h_n} + \mathbf{B}_{n+\frac{1}{2}}^T \, \mathbf{S}_{n+\frac{1}{2}} - \rho_0 \mathbf{b}_{n+\frac{1}{2}}\right] \cdot \delta_* \mathbf{u}_{n+1} \, \mathrm{d}V \\ & - \int_{\partial_t \mathscr{B}_0} \mathbf{t}_{n+\frac{1}{2}} \cdot \delta_* \mathbf{u}_{n+1} \, \mathrm{d}A - \int_{\partial_u \mathscr{B}_0} \mathbf{h}_{n+\frac{1}{2}} \cdot \delta_* \mathbf{u}_{n+1} \, \mathrm{d}A \end{aligned} \qquad (53)$$

with the linearized strain operator $\mathbf{B}_{n+\frac{1}{2}}$ defined by [8]

$$2 \, \mathbf{B}_{n+\frac{1}{2}} \, \delta_* \mathbf{u}_{n+1} := \mathbf{F}_{n+\frac{1}{2}}^T \nabla(\delta_* \mathbf{u}_{n+1}) + \nabla(\delta_* \mathbf{u}_{n+1})^T \mathbf{F}_{n+\frac{1}{2}} \qquad (54)$$

and bearing in mind the differentiation rule

$$\dot{\overline{(\bullet)}} = \frac{\mathrm{d}(\bullet)}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\mathrm{d}(\bullet)}{\mathrm{d}\alpha}\frac{1}{h_n} \tag{55}$$

as well as the *vanishing* variations $\delta_*\mathbf{u}_n$, $\delta_*\mathbf{v}_n$, $\delta_*\mathbf{p}_n$ and $\delta_*\mathbf{C}_n$ due to the initial conditions

$$\mathbf{u}(t_0) = \mathbf{u}_0 \qquad \mathbf{p}(t_0) = \rho_0\mathbf{v}_0 \tag{56}$$

$$\mathbf{v}(t_0) = \mathbf{v}_0 \qquad \tilde{\mathbf{C}}(t_0) = (\nabla\mathbf{u}_0 + \mathbf{I})^T(\nabla\mathbf{u}_0 + \mathbf{I}) \tag{57}$$

in the first time step $[t_0, t_1]$. At this point, we are able to derive spatially local relations of tensor fields at each point $\mathbf{X} \in \mathscr{B}_0$, which can be used to eliminate variables in the discrete system of equations of motion without taking spatial integrals. On the other hand, we may keep all the variables and solve a multifield formulation if we are interested in these variables for post-processing purposes, or we may eliminate the variables after the spatial integrals have been taken. The first line of Eq. (53) leads to

$$\mathbf{p}_n = \rho_0\mathbf{v}_n \qquad \mathbf{p}_{n+1} = \rho_0\mathbf{v}_{n+1} \qquad h_n\,\mathbf{v}_{n+\frac{1}{2}} = \mathbf{u}_{n+1} - \mathbf{u}_n \tag{58}$$

where Eq. (58.1) is obvious from the initial condition in Eq. (56.2), and Eq. (58.3) can be seen as first equation of motion [2]. The second line of Eq. (58) furnishes the discrete constitutive relation

$$\mathbf{S}_{n+\frac{1}{2}} = 2\,\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) + \tilde{\mathbf{S}}_{n+\frac{1}{2}} \tag{59}$$

But note that the superimposed second PIOLA-KIRCHHOFF stress tensor $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ *must not vanish* for energy consistency as in the continuous setting (compare Reference [7]). We derive it below in a separate variational problem. In the third line of Eq. (53), we take into account the symmetry of $\delta_*\mathbf{S}_{n+\frac{1}{2}}$, leading to the identity

$$\left[\mathbf{F}_{n+1}^T\mathbf{F}_n - \mathbf{F}_n^T\mathbf{F}_{n+1}\right] : \delta_*\mathbf{S}_{n+\frac{1}{2}} = 0 \tag{60}$$

and consequently to the local relation

$$\mathbf{C}_{n+1} - \mathbf{C}_n = \mathbf{F}_{n+1}^T\mathbf{F}_{n+1} - \mathbf{F}_n^T\mathbf{F}_n \Longleftrightarrow \mathbf{C}_{n+1} = \mathbf{F}_{n+1}^T\mathbf{F}_{n+1} \tag{61}$$

according to the initial condition in Eq. (57.2). Hence, we arrive at the following *variationally consistent assumed 'strain' approximation*, which is proposed for energy consistent time stepping schemes at least since the publication of Reference [6]:

$$\tilde{\mathbf{C}}_{n+\frac{1}{2}} := \frac{1}{2}\left[\mathbf{C}_n + \mathbf{C}_{n+1}\right] \tag{62}$$

The spatial approximation in the variational formulation is based on trilinear shape functions for an eight-node brick element for the volume and bilinear shape functions for an four-node quadrilateral element for the boundaries, which approximate the geometry in $\mathscr{B}_0$, the displacement vector $\mathbf{u}$ and the material velocity vector $\mathbf{v}$ at the considered discrete time points t_n. Hence, following the notation in Reference [10], we apply the approximations

$$\mathsf{u} = \mathsf{N}\,\mathbf{u} \qquad \delta_*\mathsf{u} = \mathsf{N}\,\delta_*\mathbf{u} \tag{63}$$

$$\mathsf{v} = \mathsf{N}\,\mathbf{v} \qquad \delta_*\mathsf{v} = \mathsf{N}\,\delta_*\mathbf{v} \tag{64}$$

where $\mathbf{N}$ denotes the matrix of the trilinear shape functions and the vector $\mathbf{u}$ combines the nodal displacements. An analogous notation is used for the nodal velocities and nodal variations, respectively. On the boundary $\partial\mathscr{B}_0$, we apply the approximations

$$\mathbf{u} = \bar{\mathbf{N}}\,\mathbf{u} \qquad\qquad \delta_*\mathbf{u} = \bar{\mathbf{N}}\,\delta_*\mathbf{u} \tag{65}$$

where $\bar{\mathbf{N}}$ denotes the matrix of the bilinear shape functions. The last two lines of Eq. (53) then leads to the discrete variational formulation

$$\delta_*\mathbf{u}_{n+1}^T\left[\mathbf{M}\,\frac{\mathbf{v}_{n+1}-\mathbf{v}_n}{h_n} + \int_{\mathscr{B}_0}\mathbf{B}_{n+\frac{1}{2}}^T\mathbf{S}_{n+\frac{1}{2}}\mathrm{d}V\right] = \delta_*\mathbf{u}_{n+1}^T\left[\mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}} + \mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}} + \mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}}\right] \tag{66}$$

with the system matrices

$$\mathbf{M} := \int_{\mathscr{B}_0}\rho_0\,\mathbf{N}^T\mathbf{N}\,\mathrm{d}V \qquad \mathbf{H}_t := \int_{\partial_t\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V \qquad \mathbf{H}_u := \int_{\partial_u\mathscr{B}_0}\bar{\mathbf{N}}^T\bar{\mathbf{N}}\,\mathrm{d}V \tag{67}$$

as well as the matrix representations $\mathbf{B}_{n+\frac{1}{2}}$ and $\mathbf{S}_{n+\frac{1}{2}}$ of the linearized strain operator and the second PIOLA-KIRCHHOFF stress tensor, respectively. Finally, we apply the fundamental theorem of variational calculus and arrive at the discrete system of equations of motion

$$\mathbf{M}\,\frac{\mathbf{v}_{n+1}-\mathbf{v}_n}{h_n} + \int_{\mathscr{B}_0}\mathbf{B}_{n+\frac{1}{2}}^T\mathbf{S}_{n+\frac{1}{2}} = \mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}} + \mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}} + \mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}} \tag{68}$$

If we now multiply Eq. (68) on both sides from the left by the velocity vector

$$\mathbf{v}_{n+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{v}_{n+1} + \mathbf{v}_n\right] \tag{69}$$

the first term on the left hand side takes the form

$$\begin{aligned}\mathbf{v}_{n+\frac{1}{2}}^T\mathbf{M}\,\frac{\mathbf{v}_{n+1}-\mathbf{v}_n}{h_n} &= \frac{1}{2\,h_n}\left[\mathbf{v}_{n+1}^T\mathbf{M}\,\mathbf{v}_{n+1} - \mathbf{v}_n^T\mathbf{M}\,\mathbf{v}_n\right]\\ &= \frac{\mathcal{T}_{n+1}-\mathcal{T}_n}{h_n}\end{aligned} \tag{70}$$

which denotes the discrete time derivative of the total kinetic energy. On the righthand side of Eq. (68), we obtain the discrete external power

$$\begin{aligned}\mathbf{v}_{n+\frac{1}{2}}^T\left[\mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}} + \mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}} + \mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}}\right] &= \frac{\mathbf{u}_{n+1}^T-\mathbf{u}_n^T}{h_n}\left[\mathbf{H}_t\,\mathbf{t}_{n+\frac{1}{2}} + \mathbf{H}_u\,\mathbf{h}_{n+\frac{1}{2}} + \mathbf{M}\,\mathbf{b}_{n+\frac{1}{2}}\right]\\ &= -\frac{\Pi_{n+1}^{\mathrm{ext}}-\Pi_n^{\mathrm{ext}}}{h_n}\end{aligned} \tag{71}$$

where Eq. (58.3) have been taken into account. Accordingly, we arrive at the discrete total energy balance *if the relation*

$$\begin{aligned}\int_{\mathscr{B}_0}\mathbf{v}_{n+\frac{1}{2}}\cdot\mathbf{B}_{n+\frac{1}{2}}^T\left[2\,\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_0) + \tilde{\mathbf{S}}_{n+\frac{1}{2}}\right]\mathrm{d}V &=\\ \int_{\mathscr{B}_0}\frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{h_n}\cdot\mathbf{B}_{n+\frac{1}{2}}^T\left[2\,\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_0) + \tilde{\mathbf{S}}_{n+\frac{1}{2}}\right]\mathrm{d}V &= \frac{\Pi_{n+1}^{\mathrm{int}}-\Pi_n^{\mathrm{int}}}{h_n}\end{aligned} \tag{72}$$

or

$$W_{n+1} - W_n = \left[\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) + \frac{1}{2}\,\tilde{\mathbf{S}}_{n+\frac{1}{2}} \right] : 2\,\mathbf{B}_{n+\frac{1}{2}}\,[\mathbf{u}_{n+1} - \mathbf{u}_n] \tag{73}$$

is fulfilled. On the other hand, employing the assumed 'strain' tensor in Eq. (61) in the definition of the linearized strain operator in Eq. (54), we obtain the identity

$$\begin{aligned} 2\,\mathbf{B}_{n+\frac{1}{2}}\,[\mathbf{u}_{n+1} - \mathbf{u}_n] &= \mathbf{F}^T_{n+\frac{1}{2}}\,[\mathbf{F}_{n+1} - \mathbf{F}_n] + \mathbf{F}_{n+\frac{1}{2}}\,\,[\mathbf{F}_{n+1} - \mathbf{F}_n]^T \\ &= \frac{1}{2}\left[\mathbf{C}_{n+1} - \mathbf{C}_n + \mathbf{C}^T_{n+1} - \mathbf{C}^T_n\right] \\ &= \mathbf{C}_{n+1} - \mathbf{C}_n \end{aligned} \tag{74}$$

which in the end leads to the scalar-valued constraint $\mathcal{G}(\tilde{\mathbf{S}}_{n+\frac{1}{2}})$ on the superimposed stress field $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ at each point $\mathbf{X} \in \mathscr{B}_0$, given by

$$\mathcal{G}(\tilde{\mathbf{S}}_{n+\frac{1}{2}}) := W_{n+1} - W_n - \left[\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) + \frac{1}{2}\,\tilde{\mathbf{S}}_{n+\frac{1}{2}} \right] : [\mathbf{C}_{n+1} - \mathbf{C}_n] = 0 \tag{75}$$

As the superimposed stress tensor is symmetric, $n_{\mathrm{dim}}(n_{\mathrm{dim}} + 1)/2$ components of the tensor $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ has to be uniquely determined such that the scalar-valued constraint in Eq. (75) is satisfied. Therefore, we solve the *separate constraint variational problem*

$$\delta_* \mathcal{L}(\mu, \tilde{\mathbf{S}}_{n+\frac{1}{2}}) = 0 \tag{76}$$

with

$$\mathcal{L}(\mu, \tilde{\mathbf{S}}_{n+\frac{1}{2}}) := \frac{1}{2}\,\tilde{\mathbf{C}}_{n+\frac{1}{2}}\,\tilde{\mathbf{S}}_{n+\frac{1}{2}} : \tilde{\mathbf{S}}_{n+\frac{1}{2}}\,\tilde{\mathbf{C}}_{n+\frac{1}{2}} + \mu\,\mathcal{G}(\tilde{\mathbf{S}}_{n+\frac{1}{2}}) \tag{77}$$

using the corresponding discrete EULER-LAGRANGE equations

$$\frac{\partial \mathcal{L}}{\partial \tilde{\mathbf{S}}_{n+\frac{1}{2}}} \equiv \tilde{\mathbf{C}}_{n+\frac{1}{2}}\,\tilde{\mathbf{S}}_{n+\frac{1}{2}}\,\tilde{\mathbf{C}}_{n+\frac{1}{2}} - \frac{\mu}{2}\,[\mathbf{C}_{n+1} - \mathbf{C}_n] = \mathbf{O} \qquad \frac{\partial \mathcal{L}}{\partial \mu} \equiv \mathcal{G}(\tilde{\mathbf{S}}_{n+\frac{1}{2}}) = 0 \tag{78}$$

Note that in Eq. (77) the right CAUCHY-GREEN tensor $\tilde{\mathbf{C}}_{n+\frac{1}{2}}$ operates as metric tensor as in the physically consistent deviator stress in Reference [12]. Therefore, this constraint variational problem could be also pushed forward to the current configuration $\mathscr{B}_t$, and formulated with the KIRCHHOFF stress tensor $\boldsymbol{\tau}$ and the metric $\mathbf{g}$ in $\mathscr{B}_t$. After inserting Eq. (78.1) in Eq. (78.2), we arrive at the two spatially local discrete EULER-LAGRANGE equations for the superimposed stress field $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ and the scaling factor μ at each point $\mathbf{X} \in \mathscr{B}_0$, given by

$$\tilde{\mathbf{S}}_{n+\frac{1}{2}} = \frac{\mu}{2}\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1} - \mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}} \tag{79}$$

$$2\,\mathcal{G}(\mathbf{O}) = \frac{\mu}{2}\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1} - \mathbf{C}_n] : [\mathbf{C}_{n+1} - \mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}} \tag{80}$$

We are able to search numerically for the LAGRANGE multiplier μ, but usually it is eliminated analytically. This leads to the stress tensor [8]

$$\tilde{\mathbf{S}}_{n+\frac{1}{2}} = 2\,\frac{\mathcal{G}(\mathbf{O})}{\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1} - \mathbf{C}_n] : [\mathbf{C}_{n+1} - \mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1} - \mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}} \tag{81}$$

Consequently, using the superimposed discrete stress tensor in Eq. (81), the discrete equation of motion in Eq. (68) leads, by design, to the discrete energy balance

$$\mathcal{T}_{n+1} - \mathcal{T}_n + \Pi_{n+1}^{\text{int}} - \Pi_n^{\text{int}} + \Pi_{n+1}^{\text{ext}} - \Pi_n^{\text{ext}} = 0 \iff \mathcal{H}_{n+1} = \mathcal{H}_n \tag{82}$$

which indicates exact algorithmic total energy conservation. But note that in a numerical implementation, the exact algorithmic total energy conservation is indicated by

$$|\mathcal{H}_{n+1} - \mathcal{H}_n| < \mathsf{tol} \tag{83}$$

where tol denotes the tolerance of the applied NEWTON-RAPHSON method for solving the nonlinear discrete EULER-LAGRANGE equations [11]. Further, you should be careful with solution steps where $\mathbf{C}_{n+1} \approx \mathbf{C}_n$ when applying the stress formula in Eq. (81). You should generally implement a prestep formula for the displacements $\mathbf{u}_{n+1}$ taking into account all applied loads and initial conditions [13]. If the tensor $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ is *neglected* then a TAYLOR series expansion of the strain energies W_n and W_{n+1} at the assumed 'strain' tensor $\tilde{\mathbf{C}}_{n+\frac{1}{2}}$, given by

$$\mathrm{D}W(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) : [\mathbf{C}_{n+1} - \mathbf{C}_n] = W_{n+1} - W_n + \mathcal{O}\left(\|\mathbf{C}_{n+1} - \mathbf{C}_n\|^3\right) \tag{84}$$

shows that Eq. (83) can be guaranteed only for $\|\mathbf{C}_{n+1} - \mathbf{C}_n\| < \mathsf{tol}$. Therefore, we conclude that energy consistency of the discrete EULER-LAGRANGE equations is only given if the discrete superimposed stress tensor $\tilde{\mathbf{S}}_{n+\frac{1}{2}}$ is non-vanishing.

3.2 The partitioned strain energy function

The time stepping scheme for the partitioned strain energy also follows from a time integration of the corresponding virtual power principle on the time interval $[t_0, t_N]$ of interest. Thus, we obtain a discrete variational condition at the collocation point ξ_1, given by

$$\begin{aligned} \delta_* \dot{\mathcal{H}}_d(&\mathbf{u}_{n+1}, \mathbf{v}_{n+1}, \mathbf{p}_{n+1}, \tilde{\mathbf{C}}_{n+1}, \mathbf{S}_{M_{n+\frac{1}{2}}}, \tilde{C}_{F_{n+1}}, \mathbf{S}_{F_{n+\frac{1}{2}}}; \\ &\rho_0, \boldsymbol{\kappa}_{0_F}, \boldsymbol{\kappa}_{0_M}, \mathbf{A}_0, \mathbf{b}_{n+\frac{1}{2}}, \mathbf{t}_{n+\frac{1}{2}}, \bar{\mathbf{u}}_{n+1}, \tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}}, \tilde{S}_{F_{n+\frac{1}{2}}})\, h_n = 0 \end{aligned} \tag{85}$$

The assumed squared fiber stretch $\tilde{C}_F$ is also linear piecewise continuous approximated by

$$\tilde{C}_{F_h}^n(\alpha) := C_{F_n} + \alpha\left(C_{F_{n+1}} - C_{F_n}\right) \tag{86}$$

with the 'initial' condition on the time step $[t_n, t_{n+1}]$, given by

$$C_{F_n} = \mathbf{C}_{F_n} : \mathbf{A}_0 = \mathbf{F}_{F_n}^T \mathbf{F}_{F_n} : \mathbf{A}_0 \tag{87}$$

Rearranging termes in Eq. (85) according to the independent variations $\delta_*\mathbf{p}_{n+1}$, $\delta_*\mathbf{v}_{n+1}$, $\delta_*\mathbf{u}_{n+1}$, $\delta_*\mathbf{S}_{M_{n+\frac{1}{2}}}$, $\delta_*\mathbf{C}_{n+1}$, $\delta_*\mathbf{S}_{F_{n+\frac{1}{2}}}$ and $\delta_* C_{F_{n+1}}$, we obtain the semi-discrete variational form

$$
\begin{aligned}
0 \;=\; & \int_{\mathscr{B}_0} \left[\rho_0 \mathbf{v}_{n+\frac{1}{2}} - \mathbf{p}_{n+\frac{1}{2}}\right] \cdot \delta_*\mathbf{v}_{n+1}\,\mathrm{d}V - \int_{\mathscr{B}_0} \delta_*\mathbf{p}_{n+1} \cdot \left[\mathbf{v}_{n+\frac{1}{2}} - \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{h_n}\right] \mathrm{d}V \\
& - \frac{1}{2} \int_{\mathscr{B}_0} \left[\mathbf{S}_{M_{n+\frac{1}{2}}} - 2\,\mathrm{D}W_M(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \boldsymbol{\kappa}_{0_M}) - \tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}}\right] : \delta_*\mathbf{C}_{n+1}\,\mathrm{d}V \\
& - \frac{1}{2} \int_{\mathscr{B}_0} \left[\mathbf{S}_{F_{n+\frac{1}{2}}} : \mathbf{A}_0 - 2\,\mathrm{D}\hat{W}_F(\tilde{C}_{F_{n+\frac{1}{2}}}; \boldsymbol{\kappa}_{0_F}) - \tilde{S}_{F_{n+\frac{1}{2}}}\right] : \delta_* C_{F_{n+1}}\,\mathrm{d}V \\
& - \frac{1}{2} \int_{\mathscr{B}_0} \left[\mathbf{C}_{n+1} - \mathbf{C}_n - (\mathbf{F}_{n+1} + \mathbf{F}_n)^T (\mathbf{F}_{n+1} - \mathbf{F}_n)\right] : \delta_*\mathbf{S}_{n+\frac{1}{2}}\,\mathrm{d}V \\
& - \frac{1}{2} \int_{\mathscr{B}_0} \left[C_{F_{n+1}}\mathbf{A}_0 - C_{F_n}\mathbf{A}_0 - \left(\mathbf{F}_{F_{n+1}} + \mathbf{F}_{F_n}\right)^T \left(\mathbf{F}_{F_{n+1}} - \mathbf{F}_{F_n}\right)\right] : \delta_*\mathbf{S}_{F_{n+\frac{1}{2}}}\,\mathrm{d}V \\
& + \int_{\mathscr{B}_0} \left[\frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{h_n} + \mathbf{B}^T_{n+\frac{1}{2}} \left[\mathbf{S}_{M_{n+\frac{1}{2}}} + \left(\mathbf{S}_{F_{n+\frac{1}{2}}} : \mathbf{A}_0\right)\mathbf{A}_0\right] - \rho_0 \mathbf{b}_{n+\frac{1}{2}}\right] \cdot \delta_*\mathbf{u}_{n+1}\,\mathrm{d}V \\
& - \int_{\partial_t\mathscr{B}_0} \mathbf{t}_{n+\frac{1}{2}} \cdot \delta_*\mathbf{u}_{n+1}\,\mathrm{d}A - \int_{\partial_u\mathscr{B}_0} \mathbf{h}_{n+\frac{1}{2}} \cdot \delta_*\mathbf{u}_{n+1}\,\mathrm{d}A
\end{aligned} \tag{88}
$$

The first line of Eq. (88) also furnishes the Eqs. (58). The second and third line leads to the discrete constitutive stress relations

$$
\mathbf{S}_{M_{n+\frac{1}{2}}} = 2\,\mathrm{D}W_M(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \boldsymbol{\kappa}_{0_M}) + \tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}} \tag{89}
$$

$$
\mathbf{S}_{F_{n+\frac{1}{2}}} = \left[2\,\mathrm{D}\hat{W}_F(\tilde{C}_{F_{n+\frac{1}{2}}}; \boldsymbol{\kappa}_{0_F}) + \tilde{S}_{F_{n+\frac{1}{2}}}\right] \mathbf{A}_0 \tag{90}
$$

The fourth and fivth line of Eq. (88) determines the right CAUCHY-GREEN 'strains' at the time point t_{n+1} by the equations

$$
\mathbf{C}_{n+1} = \mathbf{F}^T_{n+1}\mathbf{F}_{n+1} \qquad C_{F_{n+1}} = \mathbf{C}_{F_{n+1}} : \mathbf{A}_0 = \mathbf{F}^T_{F_{n+1}}\mathbf{F}_{F_{n+1}} : \mathbf{A}_0 \tag{91}
$$

leading to the approximations

$$
\tilde{\mathbf{C}}_{n+\frac{1}{2}} := \frac{1}{2}\left[\mathbf{C}_n + \mathbf{C}_{n+1}\right] \qquad \tilde{C}_{F_{n+\frac{1}{2}}} := \frac{1}{2}\left[C_{F_n} + C_{F_{n+1}}\right] \tag{92}
$$

Hence, the full symmetric tensor $\mathbf{C}_F$ has not to be stored at the midpoint $t_{n+\frac{1}{2}}$, but merely the scalar 'strain' $\tilde{C}_{F_{n+\frac{1}{2}}}$. Analogous to the unpartitioned case, the last lines of Eq. (88) gives the discrete system of equations of motion

$$
\begin{aligned}
\mathsf{M}\,\frac{\mathsf{v}_{n+1} - \mathsf{v}_n}{h_n} + \int_{\mathscr{B}_0} \mathsf{B}^T_{n+\frac{1}{2}} \Big[& 2\,\mathrm{D}W_M(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \boldsymbol{\kappa}_{0_M}) + 2\,\mathrm{D}\hat{W}_F(\tilde{C}_{F_{n+\frac{1}{2}}}; \boldsymbol{\kappa}_{0_F})\,\mathbf{A}_0 \\
& + \tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}} + \tilde{S}_{F_{n+\frac{1}{2}}}\,\mathbf{A}_0\Big]\,\mathrm{d}V = \mathsf{H}_t\,\mathsf{t}_{n+\frac{1}{2}} + \mathsf{H}_u\,\mathsf{h}_{n+\frac{1}{2}} + \mathsf{M}\,\mathsf{b}_{n+\frac{1}{2}}
\end{aligned} \tag{93}
$$

by taking into account the Eq. (89) and (90). Accordingly, we arrive at exact algorithmic total energy conservation in the sense of Eq. (82), if for the matrix continuum $\mathscr{B}_0^M$ the constraint

$$
\mathcal{G}_M(\tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}}) := W_{M_{n+1}} - W_{M_n} - \left[\mathrm{D}W_M(\tilde{\mathbf{C}}_{n+\frac{1}{2}}; \boldsymbol{\kappa}_{0_M}) + \frac{1}{2}\,\tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}}\right] : \left[\mathbf{C}_{n+1} - \mathbf{C}_n\right] = 0 \tag{94}
$$

is fulfilled, and *if the relation*

$$
\begin{aligned}
\int_{\mathscr{B}_0} \mathbf{v}_{n+\frac{1}{2}} \cdot \mathbf{B}^T_{n+\frac{1}{2}} \left[2\,\mathrm{D}\hat{W}_F(\tilde{C}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_{0F}) + \tilde{S}_{n+\frac{1}{2}}\right] \mathbf{A}_0 \,\mathrm{d}V &= \\
\int_{\mathscr{B}_0} \frac{\mathbf{u}_{n+1}-\mathbf{u}_n}{h_n} \cdot \mathbf{B}^T_{n+\frac{1}{2}} \left[2\,\mathrm{D}\hat{W}_F(\tilde{C}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_{0F}) + \tilde{S}_{n+\frac{1}{2}}\right] \mathbf{A}_0 \,\mathrm{d}V &= \\
\int_{\mathscr{B}_0} \left[\mathrm{D}\hat{W}_F(\tilde{C}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_{0F}) + \frac{\tilde{S}_{n+\frac{1}{2}}}{2}\right] \mathbf{A}_0 : 2\,\mathbf{B}_{n+\frac{1}{2}}\,[\mathbf{u}_{n+1}-\mathbf{u}_n]\,\mathrm{d}V &= \int_{\mathscr{B}_0}\left[\hat{W}_{F_{n+1}} - \hat{W}_{F_n}\right]\mathrm{d}V \\
\left[\mathrm{D}\hat{W}_F(\tilde{C}_{n+\frac{1}{2}};\mathbf{A}_0,\boldsymbol{\kappa}_{0F}) + \frac{\tilde{S}_{n+\frac{1}{2}}}{2}\right] \mathbf{A}_0 : [\mathbf{C}_{n+1}-\mathbf{C}_n] &= \hat{W}_{F_{n+1}} - \hat{W}_{F_n} \qquad (95)
\end{aligned}
$$

or

$$
\mathcal{G}_F(\tilde{S}_{F_{n+\frac{1}{2}}}) := \hat{W}_{F_{n+1}} - \hat{W}_{F_n} - \left[\mathrm{D}\hat{W}_F(\tilde{C}_{F_{n+\frac{1}{2}}};\boldsymbol{\kappa}_{0F}) + \frac{\tilde{S}_{F_{n+\frac{1}{2}}}}{2}\right]\left[C_{F_{n+1}} - C_{F_n}\right] = 0 \qquad (96)
$$

is fulfilled. According to Eq. (81), the superimposed stress tensor $\mathbf{S}_M$ for the matrix continuum is given by

$$
\tilde{\mathbf{S}}_{M_{n+\frac{1}{2}}} = 2\,\frac{\mathcal{G}_M(\mathbf{O})}{\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1}-\mathbf{C}_n] : [\mathbf{C}_{n+1}-\mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}}\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}}\,[\mathbf{C}_{n+1}-\mathbf{C}_n]\,\tilde{\mathbf{C}}^{-1}_{n+\frac{1}{2}} \qquad (97)
$$

The superimposed scalar stress field $\tilde{S}_F$ in the equation of motion is defined such that we have to take into account the identity

$$
\frac{\hat{W}_{F_{n+1}} - \hat{W}_{F_n}}{C_{F_{n+1}} - C_{F_n}}\,\mathbf{A}_0 = \left[\mathrm{D}\hat{W}_F(\tilde{C}_{F_{n+\frac{1}{2}}};\boldsymbol{\kappa}_{0F}) + \frac{\tilde{S}_{F_{n+\frac{1}{2}}}}{2}\right]\mathbf{A}_0 \qquad (98)
$$

which eliminates completely a FRÉCHET derivative $\mathrm{D}\hat{W}_F$ of the strain energy $\hat{W}_F$ in the equation of motion.

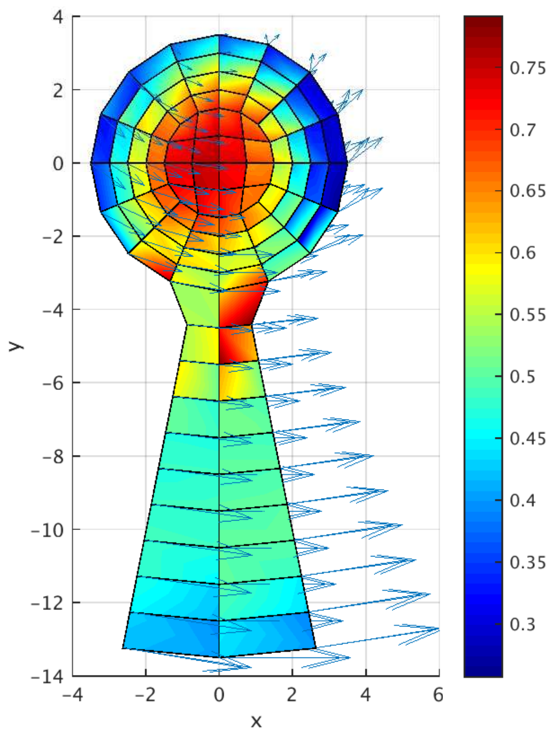
4 NUMERICAL EXAMPLE

As numerical example, we consider a transversely isotropic blade discretized in space by eight-node brick elements. In the initial configuration, the center of the blade's hub is positioned in the origin of the three-dimensional EUCLIDEAN space (see Fig. 2). The material is described by the *unpartitioned* strain energy function

$$
\hat{W}(I_1,I_2,I_3,I_4;\boldsymbol{\kappa}_0) = \hat{W}^{\mathrm{isotr}}(I_1,I_2,I_3;c_1,c_2,c_3) + \hat{W}^{\mathrm{aniso}}(I_3,I_4;c_3,c_4) \qquad (99)
$$

with the functions

$$
\begin{aligned}
\hat{W}^{\mathrm{isotr}}(I_1,I_2,I_3;c_1,c_2,c_3) &= \frac{c_1}{2}(I_3^{-\frac{1}{3}}I_1 - 3) + c_2(I_3-1)^2 \qquad (100)\\
\hat{W}^{\mathrm{aniso}}(I_3,I_4;c_3,c_4) &= \frac{c_3}{2c_4}\exp\left[c_4\left(I_3^{-1/3}I_4 - 1\right)^2 - 1\right] \qquad (101)
\end{aligned}
$$



Simulation parameter		
spatial mesh	Eight-node bricks	
element number	n_{el}	100
node number	n_{no}	238
mass density	ρ_0	1
strain energy	$W = W^{\mathrm{isotr}} + W^{\mathrm{aniso}}$	
soft material	c_1	300
	c_2	100
	c_3	240
	c_4	80
stiff material	c_1	3000
	c_2	1000
	c_3	2400
	c_4	800
fiber vector	$\mathbf{a}_0$	$[1,1,1]^T/\sqrt{3}$
initial velocity	$\mathbf{v}_0^A = \mathbf{v}_T + \boldsymbol{\omega}_0 \times \mathbf{q}_0^A$	
velocity vector	$\mathbf{v}_T$	$[2,0,-0.1]^T$
angular velocity	$\boldsymbol{\omega}_0$	$[0,0.7,0.7]^T$
NEWTON tolerance	tol	10^{-8}

Figure 2: Left: Initital configuration $\mathcal{B}_0$ of the fiber-reinforced blade. The colours indicate the VON MISES stress at the temporal Gauss point $t_{0+1/2}$ determined by the cG(1) method in the non-stiff case. The arrows show the initial velocity field of the free flight. Right: Simulation parameter of the motion and of the algorithm.

the parameters c_1, c_2, c_3 and the dimensionless parameter c_4 (compare Reference [14]). The applied material parameter values are summarized in Fig. 2 on the right. We distinguish between soft and stiff material. The blade are in free flight due to its initial translational velocity field and its initial angular velocity field (see Fig. 2). We compare two numerical methods:

(i) the variational consistent discrete method presented above, referred to as eG(1) method in the following, and

(ii) the continuous Galerkin cG(1) method or midpoint rule, respectively, given by Eq. (68) and the corresponding second PIOLA-KIRCHHOFF stress tensor

$$\mathbf{S}_{n+\frac{1}{2}}^{\mathrm{mid}} = 2\,\mathrm{D}W(\mathbf{F}_{n+\frac{1}{2}}^T \mathbf{F}_{n+\frac{1}{2}}; \mathbf{A}_0, \boldsymbol{\kappa}_0) \tag{102}$$

based on a *temporally discontinuous 'strain' approximation*.

Considering soft material, both methods show similar current configurations for a moderate constant time step size. Therefore, we show only the motion of the cG(1) method in Fig. 3. But, by changing the time step size during the simulation, the NEWTON-RAPHSON method in the time loop of the cG(1) method aborts after some time steps. This can be shown by plotting the total energy of the blade versus time (see Fig. 4 on the left). In contrast to the eG(1) method, the cG(1) method shows an oscillating total energy with an energy blow-up after the time step size change. Considering stiff material, no time step size change is necessary for illustrating the unstable behaviour of the cG(1) method in contrast to the eG(1) method (see Fig. 4 on the right).

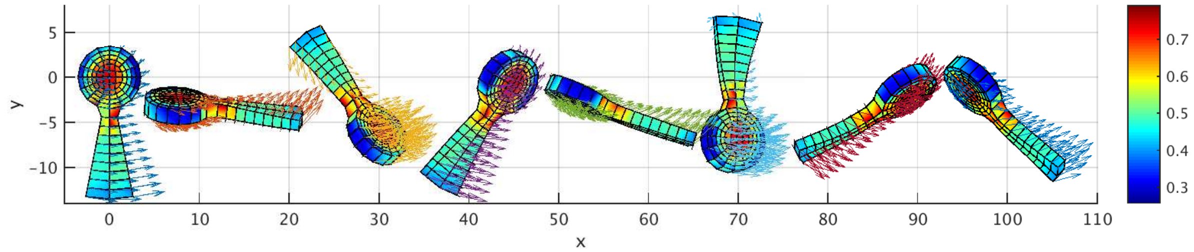


Figure 3: Current configurations $\mathcal{B}_{t_n}$ of the blade with the non-stiff material determined by the cG(1) method, starting at $t_0 = 0$ on the left. The colour indicates the VON MISES stress at the temporal Gauss point $t_{n+1/2}$. The arrows shows the current Lagrangian velocity field.

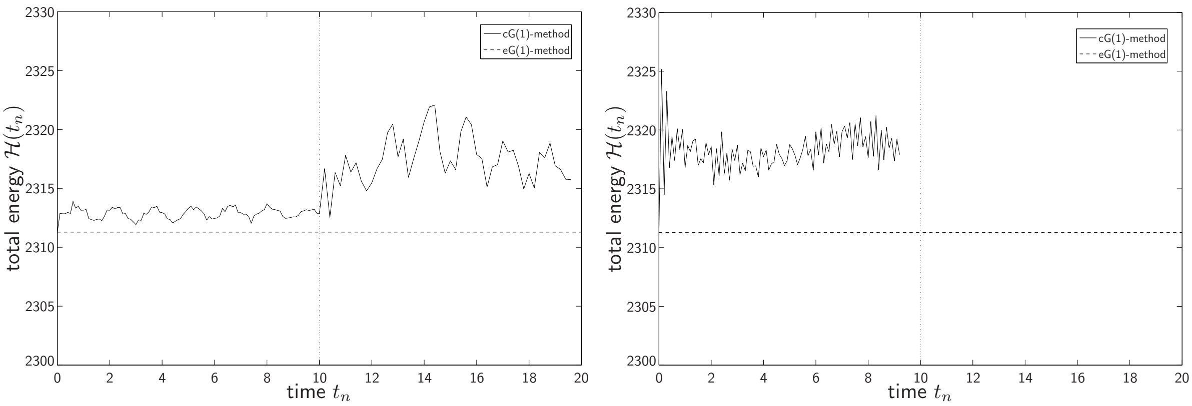


Figure 4: Comparison of the total energy $\mathcal{H}_n$ of the cG(1)-method and the eG(1)-method, respectively, using the soft material (left) and stiff material (right). The time step size is 0.1 for $t \leq 10$ and 0.2 for $t > 10$.

5 SUMMARY

In this paper, we consider transversely isotropic materials from two perspectives. We examine

1. the general case of formally one free energy function with no separation of tensor invariants (*unpartitioned free energy function*), and

2. a partition of the free energy function into two separate terms corresponding to isotropic and anisotropic invariants, respectively (*partitioned free energy function*).

The reason is that the fundamental theorem of calculus corresponding to partitioned free energy functions can be split into separate equations as it is well-known from the kinetic and potential energy of natural systems with respect to inertial reference frames. As the fundamental theorem of calculus serves as a design criterion for energy consistent time stepping algorithms, a separation into two criteria therefore allows to modify the algorithm in a more targeted manner. Further, already implemented energy consistent time stepping algorithms for isotropic materials could be extented rather than modified to transversely isotropic materials or composite materials with more than one family of fibers.

In this work, we start the design of energy consistent time stepping algorithms by *discretizing a mixed variational principle*, because we aim at a unified design procedure for these important algorithms. Such an unified procedure already exists for momentum consistent time stepping algorithms leading to the so-called variational integrators [5]. In this variational framework, the consistency with the momentum balances does not depend on the numerical quadrature as in usual finite difference or GALERKIN-based schemes. Energy consistent time stepping algorithms, however, are hitherto designed for specific mechanical problems [2, 3, 8, 11, 13], although the corresponding design procedures exhibit many common features. In each of these references, the discrete total energy balance is satisfied by introducing

- temporally *continuous* approximations of the independent argument tensors of the total energy, together with
- temporally discontinuous superimposed work conjugate tensor fields emanating from spatially local formulations of the fundamental theorem of calculus in time.

But, the application of this concept for designing higher-order accurate time stepping schemes as generalisation of existing second-order accurate schemes raises questions in the details [2, 3], starting with the temporally continuous approximation of the independent argument tensors of the *strain energy*. In these references, a mixture of 'strain' tensor approximations has to be used for satisfying energy consistency, which is not obvious from a physical perspective. These problems and the need of a unified framework have led to the herewith presented idea of discretizing a mixed variational principle, providing

1. a *proof of existing adhoc time approximations*, and
2. *new higher-order accurate* energy consistent time approximations (see Appendix).

The first adhoc time approximation is the midpoint evaluation of the GREEN-LAGRANGE strain tensor in Reference [6], or later called *assumed 'strain' approximation* in Reference [15], respectively. This approximation is often used as a physically modivated assumption (frame-invariance of discrete strains), or as an inherent part of energy consistent discrete gradients of strain energy functions in finite difference schemes [8]. In Reference [6], this temporally continuous approximation of the GREEN-LAGRANGE strain tensor is modivated by the exact quadrature of the approximated strain energy function pertaining to the quadratic SAINT-VENANT KIRCHHOFF model. The discrete total energy balance corresponding to this strain energy function is therefore fulfilled without a superimposed stress field, or in other words, a superimposed stress field vanish for this strain energy function as in the temporally continuous equations of motion. Hence, inspired by three-field variational functionals of EAS methods [16], we here introduce a temporally continuous strain tensor with the corresponding natural ordinary differential equation in time by means of a mixed variational principle. In this way, we actually prove the *variational consistency of the assumed 'strain' approximation* for well-known second-order accurate methods, and derive a *new* assumed 'strain' approximation for higher-order accurate schemes which avoids unphysical approximation mixtures as in Reference [3] (see Appendix).

The second adhoc time approximation is the superimposed stress tensor in References [2, 3], based on the well-known discrete gradient in Reference [17]. This superimposed stress tensor is derived from a constraint variational problem at each point $\mathbf{X} \in \mathscr{B}_0$ in References [3], which therefore provides a proof of the uniqueness of this superimposed stress tensor. However, this variational problem is *not physically modivated* and therefore not invariant with respect to a push-forward in the spatial configuration $\mathcal{B}_t$. This problem has been caused by not using a coordinate free and metric independent geometric formulation of continuum mechanics. In fact, the Euclidean metric δ_{AB} is assumed in the reference configuration from the outset. Therefore, in this paper, we arrive at the right CAUCHY-GREEN tensor as metric tensor by using a covariant tensor formulation. The obtained constraint variational problem is therefore invariant under a push-forward in the spatial configuration $\mathcal{B}_t$, and leads to an equivalent variational problem with respect to the KIRCHHOFF stress tensor. As special case for a linear approximation in time, we obtain the superimposed stress field in Reference [8]. Note, however, that a further improvement of the computational performance in comparison to the superimposed stress tensor in References [3] is not recognized by considering free flights of stiff materials. *But if the algorithm has to be pushed forward in a computational more efficient spatial setting, the new superimposed stress field is necessary.* Further note that the constraint variational problems are still separate variational problems of parameters of the mixed variational principle.

A APPENDIX

In this appendix, we show an interesting consequence of the above theory for the new assumed 'strain' approximation of higher-order accurate time integration schemes, i.e. schemes which take into account inner time points $t_{n+\alpha_i}$, $\alpha_i \in]0,1[$, beside the time step boundary points t_n and t_{n+1}. This inner time points are usually equidistant distributed over the time step (see Table 1). Here, we have to start in the unpartitioned case with the discrete principle

$$\sum_{n=0}^{N-1}\sum_{i=1}^{k} \delta_* \dot{\mathcal{H}}(\dot{\mathbf{u}}_h^n(\xi_i), \dot{\mathbf{v}}_h^n(\xi_i), \dot{\mathbf{p}}_h^n(\xi_i), \dot{\tilde{\mathbf{C}}}_h^n(\xi_i), \mathbf{S}_h^n(\xi_i); \\ \rho_0, \mathbf{A}_0, \boldsymbol{\kappa}_0, \mathbf{b}_h^n(\xi_i), \mathbf{t}_h^n(\xi_i), \dot{\bar{\mathbf{u}}}_h^n(\xi_i), \tilde{\mathbf{S}}_h^n(\xi_i))\, w_i\, h_n = 0 \tag{103}$$

where ξ_i, w_i, $i = 1,\dots,k$, denote the quadrature points and weights, respectively, and k the degree of the shape functions $M_j(\alpha)$, $j = 1,\dots,k+1$ in time. Usually, the Lagrangian shape functions and the Gaussian quadrature rules are used (see Table 1 and Table 2, respectively). According to this principle, we obtain the weak equation

$$\sum_{i=1}^{k} \int_{\mathcal{B}_0} \delta_* \mathbf{S}_h^n(\xi_i) : \left[\frac{\mathrm{d}\tilde{\mathbf{C}}_h^n(\xi_i)}{\mathrm{d}\alpha} - \overset{\circ}{\mathbf{C}}(\overset{\circ}{\mathbf{u}}{}_h^n(\xi_i)) \right] w_i \, \mathrm{d}V = 0 \tag{104}$$

with the assumed 'strain' approximation

$$\frac{\mathrm{d}\tilde{\mathbf{C}}_h^n(\alpha)}{\mathrm{d}\alpha} = \sum_{j=1}^{k+1} \overset{\circ}{M}_{j+1}(\alpha)\, \mathbf{C}_j^n \equiv \sum_{i=1}^{k} \tilde{M}_i(\alpha)\, \tilde{\mathbf{C}}_i^n \tag{105}$$

and the shorthand notation $\overset{\circ}{(\bullet)}$ for the differentiation with respect to α. The tensors $\mathbf{C}_j^n$ and $\tilde{\mathbf{C}}_i^n$ designate the independend nodal values of the assumed 'strain' approximation and its derivative, respectively, at the corresponding time points α_i and $\tilde{\alpha}_i$ (see Table 1). Having again an elimination of the assumed 'strain' field in mind, we arrive at the spatially local relation

$$\sum_{j=1}^{k+1} \overset{\circ}{M}_{j+1}(\xi_i)\, \mathbf{C}_j^n - \overset{\circ}{\mathbf{C}}(\overset{\circ}{\mathbf{u}}{}_h^n(\xi_i)) = \mathbf{O} \qquad (i = 1,\dots,k) \tag{106}$$

After further algebraic transformations, the unkonwn nodal values $\mathbf{C}_l^n$, $l = 2,\dots,k$, becomes

$$\mathbf{C}_l^n := \sum_{i=1}^{k} m_{li}\, \overset{\circ}{\mathbf{C}}(\overset{\circ}{\mathbf{u}}{}_h^n(\xi_i)) + \mathbf{C}_1^n \tag{107}$$

by taking into account the initial condition $\mathbf{C}_1^n := (\mathbf{F}_1^n)^T \mathbf{F}_1^n \equiv \mathbf{C}_n$. The coefficients A_{li} are the components of the $k \times k$ matrix

$$\mathbf{m} = \begin{bmatrix} \overset{\circ}{M}_2(\xi_1) & \dots & \overset{\circ}{M}_{k+1}(\xi_1) \\ \vdots & \dots & \vdots \\ \overset{\circ}{M}_2(\xi_k) & \dots & \overset{\circ}{M}_{k+1}(\xi_k) \end{bmatrix}^{-1} \tag{108}$$

In the case of linear time approximations ($k = 1$), these relations lead to the nodal values

$$\mathbf{C}_n \equiv \mathbf{C}_1^n = (\mathbf{F}_1^n)^T \mathbf{F}_1^n \qquad\qquad \mathbf{C}_{n+1} \equiv \mathbf{C}_2^n = (\mathbf{F}_2^n)^T \mathbf{F}_2^n \tag{109}$$

(compare Eq. (61). For quadratic time approximations ($k = 2$), we arrive at the nodal values

$$\mathbf{C}_n \equiv \mathbf{C}_1^n := (\mathbf{F}_1^n)^T \mathbf{F}_1^n \tag{110}$$

$$\mathbf{C}_2^n := \frac{1}{3}\left[\frac{\mathbf{F}_1^n + \mathbf{F}_3^n}{2} - \mathbf{F}_2^n\right]^T \left[\frac{\mathbf{F}_1^n + \mathbf{F}_3^n}{2} - \mathbf{F}_2^n\right] + (\mathbf{F}_2^n)^T \mathbf{F}_2^n \tag{111}$$

$$\mathbf{C}_{n+1} \equiv \mathbf{C}_3^n := (\mathbf{F}_3^n)^T \mathbf{F}_3^n \tag{112}$$

For higher degrees of shape functions, we obtain analogous results. Accordingly, the nodal values at the time step boundaries t_n and t_{n+1} are solely determined by $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$, respectively, but the nodal values at the inner time points depends on the *displacements of all time points*. *This is in contrast* to the (frame-indifferent) assumed 'strain' approximation $\bar{\mathbf{C}}_h^n(\alpha)$ defined in Reference [3] by the extrapolation

$$\bar{\mathbf{C}}_h^n(\alpha) = \sum_{j=1}^{k+1} M_j(\alpha)\,(\mathbf{F}_j^n)^T \mathbf{F}_j^n \tag{113}$$

of the formula $\tilde{\mathbf{C}}_h^n(\alpha)$ for linear time approximations ($k = 1$) known from Reference [15]. As the first term in Eq. (111) is unknowingly neglected in Eq. (113), the authors of Reference [3] have to introduce a mixture of time approximations in the superimposed stress tensor for energy consistency.

On the other hand, looking at Eq. (111), we may recognize a possibility to simplify the relations for the nodal values $\mathbf{C}_2, \dots \mathbf{C}_{k-1}$ by introducing an *assumed deformation gradient field*, which for $k = 2$ takes the form

$$\tilde{\mathbf{F}}_h^n(\alpha) = M_1(\alpha)\,\mathbf{F}_1^n + M_2(\alpha)\,\hat{\mathbf{F}}_2^n + M_3(\alpha)\,\mathbf{F}_3^n \tag{114}$$

with the nodal value

$$\hat{\mathbf{F}}_2^n := \frac{\mathbf{F}_1^n + \mathbf{F}_3^n}{2} \neq \nabla \mathbf{u}_2^n + \mathbf{I} \equiv \nabla \mathbf{u}_{n+\frac{1}{2}} + \mathbf{I} \tag{115}$$

Thus, the approximated displacement field $\mathbf{u}_h^n(\alpha)$ is here connected to the deformation gradient field only at the boundaries of the time step $[t_n, t_{n+1}]$ with the linear approximation

$$\tilde{\mathbf{F}}_h^n(\alpha) = \tilde{M}_1(\alpha)\,\mathbf{F}_1^n + \tilde{M}_2(\alpha)\,\mathbf{F}_3^n \tag{116}$$

The corresponding assumed 'strain' field reads

$$\begin{aligned} \tilde{\mathbf{C}}_h^n(\alpha) &= M_1(\alpha)\,(\mathbf{F}_1^n)^T \mathbf{F}_1^n + M_2(\alpha)\,(\hat{\mathbf{F}}_2^n)^T \hat{\mathbf{F}}_2^n + M_3(\alpha)\,(\mathbf{F}_3^n)^T \mathbf{F}_3^n && (117) \\ &= (\alpha - 1)^2\,(\mathbf{F}_1^n)^T \mathbf{F}_1^n + \alpha(\alpha - 1)\left[(\mathbf{F}_1^n)^T \mathbf{F}_3^n + (\mathbf{F}_3^n)^T \mathbf{F}_1^n\right] + \alpha^2\,(\mathbf{F}_3^n)^T \mathbf{F}_3^n && (118) \end{aligned}$$

However, we have to examine this possibility with respect to the accuracy order of the resulting time integration scheme. Further, a linear time approximation of the deformation gradient field and a quadratic time approximation of the displacement field seems to be inconsistent, in contrast to an analogous space approximation [14].

Nevertheless, the accuracy order of the time integration scheme corresponding to the approximation $\tilde{\mathbf{C}}_h^n(\alpha)$ with the nodal values in Eq. (107) is guaranteed for shape functions of degree up to four ($k = 4$). We have even implemented this approximation in the finite element code associated with the thermo-mechanical problem in Reference [13], and obtained the numerical results in Fig. 5. This approximation is therefore even recommended for mechanically coupled problems. The higher-order approximation of the superimposed stress tensor of the unpartitioned strain energy function at the temporal Gauss point is given by

$$\tilde{\mathbf{S}}_h^n(\xi_i) := 2\,\frac{\mathcal{G}(\mathbf{O})}{\displaystyle\sum_{l=1}^{k}[\tilde{\mathbf{C}}_h^n(\xi_l)]^{-1}\,\overset{\circ}{\overline{\tilde{\mathbf{C}}_h^n}}(\xi_l) : \overset{\circ}{\overline{\tilde{\mathbf{C}}_h^n}}(\xi_l)\,[\tilde{\mathbf{C}}_h^n(\xi_l)]^{-1}\,w_l}\,[\tilde{\mathbf{C}}_h^n(\xi_i)]^{-1}\,\overset{\circ}{\overline{\tilde{\mathbf{C}}_h^n}}(\xi_i)\,[\tilde{\mathbf{C}}_h^n(\xi_i)]^{-1} \tag{119}$$

with

$$\mathcal{G}(\mathbf{O}) := W_{n+1} - W_n - \sum_{l=1}^{k} \mathrm{D}W(\tilde{\mathbf{C}}_h^n(\xi_l);\mathbf{A}_0,\boldsymbol{\kappa}_0) : \overset{\circ}{\overline{\tilde{\mathbf{C}}_h^n}}(\xi_l)\,w_l = 0 \tag{120}$$

The superimposed stress $\tilde{S}_F$ corresponding to the partitioned strain energy is due to the scalar-valued argument $\tilde{C}_F$ analogous to the dynamical problem of a particle system in Reference [3] (compare Eq. (98) with the case $k = 1$). Hence, we obtain the relation

$$\tilde{S}_{F_h}^n(\xi_i) := 2\,\frac{\mathcal{G}(0)}{\displaystyle\sum_{l=1}^{k}\overset{\circ}{\overline{\tilde{C}_{F_h}^n}}(\xi_l)\;\overset{\circ}{\overline{\tilde{C}_{F_h}^n}}(\xi_l)\,w_l}\;\overset{\circ}{\overline{\tilde{C}_{F_h}^n}}(\xi_i) \tag{121}$$

with

$$\mathcal{G}_F(0) := \hat{W}_{F_{n+1}} - \hat{W}_{F_n} - \sum_{l=1}^{k} \mathrm{D}\hat{W}_F(\tilde{C}_{F_h}^n(\xi_l);\boldsymbol{\kappa}_{0_F})\;\overset{\circ}{\overline{\tilde{C}_{F_h}^n}}(\xi_l)\,w_l = 0 \tag{122}$$

and the fiber assumed strain approximation

$$\tilde{C}_{F_h}^n(\alpha) = \sum_{j=1}^{k+1} M_{j+1}(\alpha)\,C_{F_j}^n \tag{123}$$

where the nodal values $C_{F_l}^n$, $l = 2,\ldots,k$, take the form

$$C_{F_l}^n := \sum_{i=1}^{k} A_{li}\,\overset{\circ}{\mathbf{C}}(\overset{\circ}{\mathbf{u}}{}_h^n(\xi_i)) : \mathbf{A}_0 + \mathbf{C}_1^n : \mathbf{A}_0 \tag{124}$$

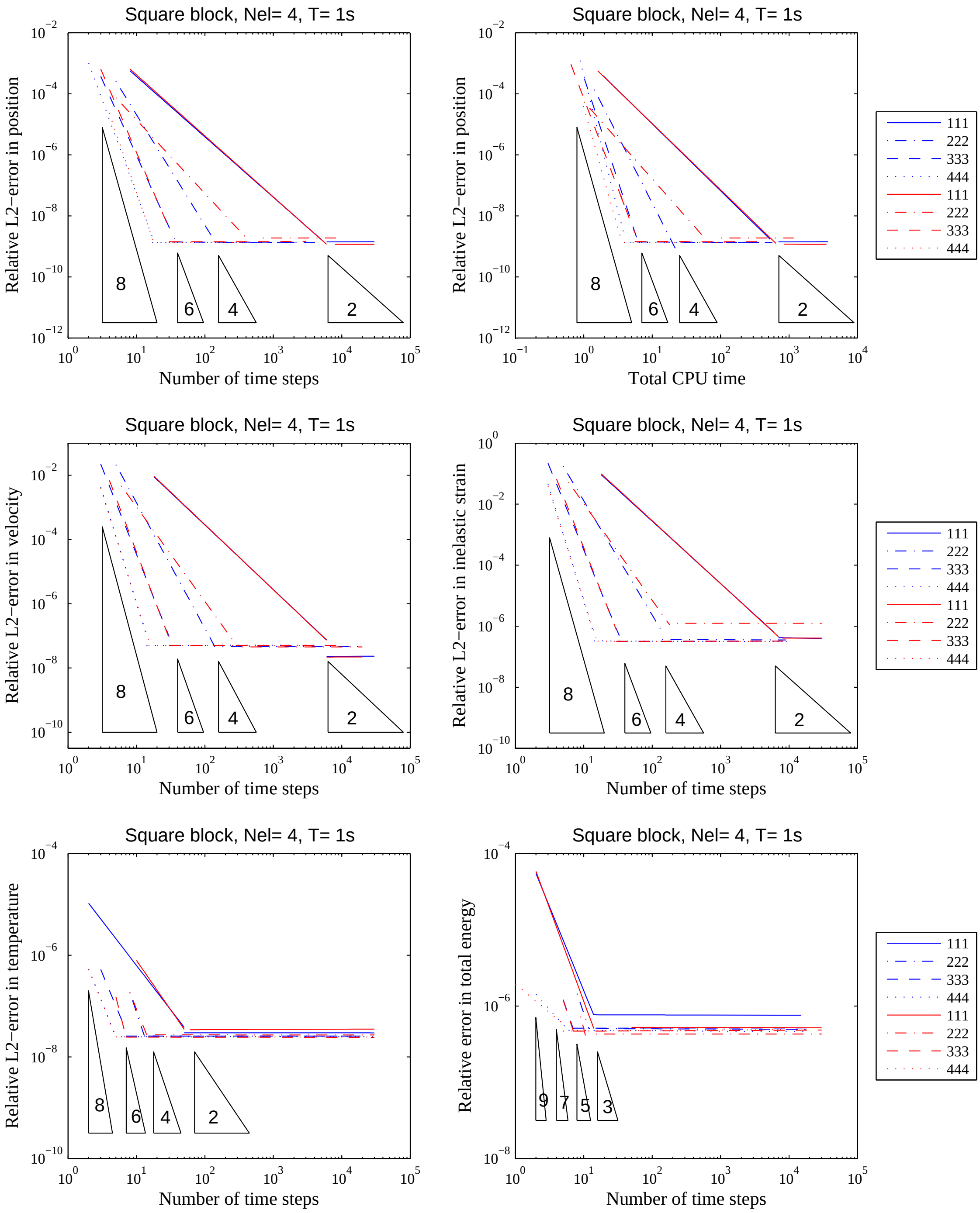


Figure 5: Accuracy orders of the energy consistent time stepping scheme presented above using the new assumed 'strain' approximation, determined with a flying stiff square discretized by four four-node quadrilateral elements. We investigated the thermo-mechanically problem in Reference [13]. The plots show the relative L_2 errors at the final simulation time $T = 1$. For shape functions of degree k in the mechanical and thermal fields (labels 'kkk'), we obtain the accuracy order $2k$. The order of the total energy has the order $2k + 1$, because the temperature is calculated with an energy consistent discontinuous GALERKIN method. Through the strong thermo-mechanical coupling, the temperature shows the same order as the displacements or current positions, respectively.

Table 1: Lagrangian shape functions in time of degree k and $k-1$ with respect to the parent domain $[0,1]$.

k	$M_j(\alpha)$	α_j	$\tilde{M}_i(\alpha)$	$\tilde{\alpha}_i$
1	$1-\alpha$ α	0 1	1	1
2	$(2\alpha-1)(\alpha-1)$ $-4\alpha\,(\alpha-1)$ $(2\alpha-1)\,\alpha$	0 $\frac{1}{2}$ 1	$1-\alpha$ α	0 1
3	$-\frac{9}{2}\,(\alpha-\frac{1}{3})(\alpha-\frac{2}{3})(\alpha-1)$ $\frac{27}{2}\,(\alpha-\frac{2}{3})(\alpha-1)\,\alpha$ $-\frac{27}{2}\,(\alpha-\frac{1}{3})(\alpha-1)\,\alpha$ $\frac{9}{2}\,(\alpha-\frac{1}{3})(\alpha-\frac{2}{3})\,\alpha$	0 $\frac{1}{3}$ $\frac{2}{3}$ 1	$(2\alpha-1)(\alpha-1)$ $-4\alpha\,(\alpha-1)$ $(2\alpha-1)\,\alpha$	0 $\frac{1}{2}$ 1
4	$\frac{32}{3}\,(\alpha-\frac{1}{4})(\alpha-\frac{1}{2})(\alpha-\frac{3}{4})(\alpha-1)$ $-\frac{128}{3}\,(\alpha-\frac{1}{2})(\alpha-\frac{3}{4})(\alpha-1)\alpha$ $64\,(\alpha-\frac{1}{4})(\alpha-\frac{3}{4})(\alpha-1)\alpha$ $-\frac{128}{3}\,(\alpha-\frac{1}{4})(\alpha-\frac{1}{2})(\alpha-1)\alpha$ $\frac{32}{3}\,(\alpha-\frac{1}{4})(\alpha-\frac{1}{2})(\alpha-\frac{3}{4})\alpha$	0 $\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$ 1	$-\frac{9}{2}\,(\alpha-\frac{1}{3})(\alpha-\frac{2}{3})(\alpha-1)$ $\frac{27}{2}\,(\alpha-\frac{2}{3})(\alpha-1)\,\alpha$ $-\frac{27}{2}\,(\alpha-\frac{1}{3})(\alpha-1)\,\alpha$ $\frac{9}{2}\,(\alpha-\frac{1}{3})(\alpha-\frac{2}{3})\,\alpha$	0 $\frac{1}{3}$ $\frac{2}{3}$ 1

Table 2: Gaussian quadrature with N_{qp} Gauss points with respect to the temporal parent domain $[0,1]$.

N_{qp}	ξ_l	w_l
1	$1/2$	1
2	$(1-1/\sqrt{3})/2$ $(1+1/\sqrt{3})/2$	$1/2$ $1/2$
3	$(1-\sqrt{3/5})/2$ $1/2$ $(1+\sqrt{3/5})/2$	$5/18$ $4/9$ $5/18$
4	$(1-\sqrt{3/7+2\,\sqrt{6/5}/7})/2$ $(1-\sqrt{3/7-2\,\sqrt{6/5}/7})/2$ $(1+\sqrt{3/7-2\,\sqrt{6/5}/7})/2$ $(1+\sqrt{3/7+2\,\sqrt{6/5}/7})/2$	$(3-\sqrt{5/6})/12$ $1/2-(3-\sqrt{5/6})/12$ $1/2-(3-\sqrt{5/6})/12$ $(3-\sqrt{5/6})/12$

REFERENCES

[1] Kaliske M (2000) A formulation of elasticity and viscoelasticity for fibre reinforced material at small and finite strains. Comput Methods Appl Mech Engrg 185:225–243.

[2] Erler N and Groß M (2015) Energy-momentum conserving higher-order time integration of nonlinear dynamics of finite elastic fiber-reinforced continua. Computational Mechanics 55(5):921–942.

[3] Groß M and Betsch P and Steinmann P (2005) Conservation properties of a time FE method. Part IV: higher order energy and momentum conserving schemes. Int J Numer Methods Engng 63:1849–1897.

[4] Schröder J and Neff P and Balzani D (2005) A variational approach for materially stable anisotropic hyperelasticity. Int J Sol Struc 42:4352–4371.

[5] Schlögl T. and Leyendecker S. (2016) Electrostatic-viscoelastic finite element model of dielectric actuators. Comput Methods Appl Mech Engrg 299:421–439.

[6] Simo JC and Tarnow N (1992) The Discrete Energy-Momentum Method. Conserving Algorithms for Nonlinear Elastodynamics. Z angew Math Phys 43:757–792.

[7] Betsch P and Janz A (2016) An energymomentum consistent method for transient simulations with mixed finite elements developed in the framework of geometrically exact shells. Int J Numer Methods Engng, early view, DOI: 10.1002/nme.5217.

[8] Armero F and Zambrana-Rojas C (2007) Volume-preserving energy-momentum schemes for isochoric multiplicative plasticity. Comput Methods Appl Mech Engrg 196:4130–4159.

[9] Schröder B and Kuhl D (2015) Small strain plasticity: classical versus multifield formulation. Archive of Applied Mechanics 85(8):1127–1145.

[10] Schröder J, Wriggers P and Balzani D (2011) A new mixed finite element based on different approximations of the minors of deformation tensors. Comput Methods Appl Mech Engrg 200:3583–3600.

[11] Betsch P and Steinmann P (2000) Inherently Energy Conserving Time Finite Elements for Classical Mechanics. Journal of Computational Physics 160:88–116.

[12] Simo JC (1987) On a Fully Three-Dimensional Finite-Strain Viscoelastic Damage Model: Formulation and Computational Aspects. Comput Methods Appl Mech Engrg 60:153–173.

[13] Groß M and Betsch P (2011) Galerkin-based energy-momentum consistent time-stepping algorithms for classical nonlinear thermo-elastodynamics. Math Comp Sim 82:718–770.

[14] Schröder J, Neff P, Balzani D (2005) A variational approach for materially stable anisotropic hyperelasticity. Int J Solid Struct 42:4352–4371.

[15] Betsch P and Steinmann P (2001) Conservation properties of a time FE method. Part II: Time-Stepping Schemes for Nonlinear Elastodynamics. Int J Numer Methods Engng 50:1931–1955.

[16] Klinkel S and Wagner W (1997) A geometrical non-linear brick element based on the EAS-method. Int J Numer Methods Engng 40:4529–4545.

[17] Gonzalez O (2000) Exact Energy and Momentum Conserving Algorithms for General Models in Nonlinear Elasticity. Comput Methods Appl Mech Engrg 190:1763–1783.