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DISCONTINUOUS GALERKIN METHOD WITH REDUCED INTEGRATION SCHEME FOR THE BOUNDARY TERMS IN ALMOST INCOMPRESSIBLE LINEAR ELASTICITY

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Abstract.

A problem of incompressibility is explored in this paper with application of a discontinuous Galerkin (dG) method with reduced integration. We apply two reduced integration schemes, namely fully and mixed reduced integrations for the Incomplete Interior Penalty Galerkin (IIPG) class of the dG method. The numerical results show convergent solutions with respect to a sufficiently large value of the penalty term and number of elements. Additionally, a comparison between the standard continuous Galerkin (cG) method and the dG method are established to compare and contrast the behavior. Finally, the dG method shows a faster convergence with respect to the number of elements.

1 INTRODUCTION

Recent advancements in the field of computational engineering have enabled scientists to define novel discretization schemes in solid mechanics. One of these schemes, discussed in this paper is the discontinuous Galerkin (dG) method. Unlike many standard Galerkin methods, the dG method is a non-conforming finite element method. Using the dG method, one obtains discontinuities between the interior element boundaries. This is achieved through a weak enforcement of the continuity on displacement. To this end, one must modify the weak form in a way that the integration by parts is applied not only on the domain boundaries, but also additionally, on the subdomain boundaries [1].

The discontinuous Galerkin method was initially introduced as a new classification in finite element methods by Reed and Hill [2] to solve a problem of a nuclear transport partial differential equation in 1971. Later, Baker [3] applied a dG method with some modifications for elliptic problems - refer to [4] and [5] for more details. Nitsche [6] contributed to dG a penalty term on the internal subdomain boundaries in order to stabilize the solution.

Although, dG was initially used in fluid mechanics [7], it found its way to solid mechanics as well (e.g. [8]) to remedy frequently encountered problems. An application of this method is seen in elliptic problems like incompressibility, which results in the well-known volumetric locking phenomenon or in some shell (plate) constructions with shear locking problems [8]. Hansbo und Larson [9] investigated the locking-free behavior of the dG method for (near) incompressibility with triangle meshes. Another application of the dG method in solid mechanics can be found in the work of Mergheim et al. [10]. They applied dG elements in the pre-failure regime to avoid stress oscillations just before the failure.

In this paper, we investigated the reduced integration method for certain boundary terms to make this method more efficient and reduce the time of calculation.

The present work is organized as follows: the first chapter introduces the governing equations of the dG method. Then the numerical integration scheme and the reductions, which are applied in order to decrease the calculation time, are clarified. Finally, a benchmark example is simulated to evaluate the method and investigate the differences between the continuous and discontinuous Galerkin methods.

2 GOVERNING EQUATIONS OF DG

The dG method has different variations depending on the extended terms of its weak form. In our case, we apply the Incomplete Interior Penalty Galerkin (IIPG). This method is non-symmetric due the absence of the symmetric term of dG method and contains the stabilization term, namely penalty term.

2.1 Strong form

The strong form of the equilibrium of the forces in standard FEM and the boundary conditions are given by

$$\begin{aligned} -div(\boldsymbol{\sigma}) &= \boldsymbol{f} \qquad \text{in} \qquad \mathcal{B}, \\ \boldsymbol{u} &= \boldsymbol{u}^p \qquad \text{on} \qquad \partial\mathcal{B}_u, \\ \boldsymbol{\sigma}\,\boldsymbol{n}_e &= \boldsymbol{t}^p \qquad \text{on} \qquad \partial\mathcal{B}_t. \end{aligned} \tag{1}$$

Where $\boldsymbol{\sigma} = \boldsymbol{C}:\boldsymbol{\varepsilon}$ is the Cauchy stress tensor with $\boldsymbol{C}$ as the forth order stiffness tensor, e.g. elasticity module E and Poisson's ratio ν and $\boldsymbol{\varepsilon}$ as the symmetric strain tensor, which is in fact the symmetric part of the gradient of the displacement field $\boldsymbol{u}$. Additionally, $\boldsymbol{f}$ represents the body forces, $\boldsymbol{u}^p$ and $\boldsymbol{t}^p$ are the prescribed displacement and prescribed traction on the Drichlet boundary $\partial\mathcal{B}_u$ and Neumann boundary $\partial\mathcal{B}_t$, respectively.

In the dG method we additionally introduce discontinuities along the internal boundaries Γ of the body $\mathcal{B}$. This divides the body into to $\mathcal{B}^+$ and $\mathcal{B}^-$ parts, with the normal vector $\hat{\boldsymbol{n}}$ directing from the negative side to the positive side, see Figure 1.

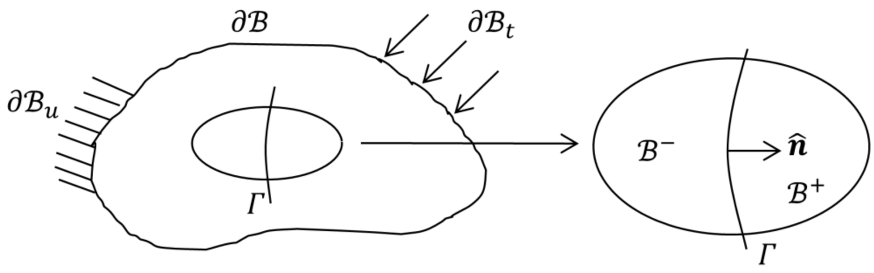

Figure 1: Discontinuity Γ within the body $\mathcal{B}$.

The jump$[\![\blacksquare]\!]$ and average $\{\blacksquare\}$ of quantities are defined as follows:

$$[\![\boldsymbol{u}]\!] = \boldsymbol{u}^+|_\Gamma - \boldsymbol{u}^-|_\Gamma,$$

$$\{\boldsymbol{u}\} = \frac{1}{2}\,(\boldsymbol{u}^+|_\Gamma + \boldsymbol{u}^-|_\Gamma). \tag{2}$$

Introducing the dG method to the strong form, we need to impose extra conditions on the internal boundaries Γ. On these boundaries, the continuity of the displacements field and the traction vector is prescribed by the exact solution:

$$[\![\boldsymbol{u}]\!] = 0$$

$$[\![\boldsymbol{\sigma}\,\hat{\boldsymbol{n}}]\!] = 0. \tag{3}$$

2.2 Weak form

The weak form of dG is obtained by integration by parts on the internal subdomains. Nonetheless, the penalty term must be added to stabilize the solution [6]:

$$\int_{\mathcal{B}^+\cup\mathcal{B}^-} \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\; dV + \int_\Gamma [\![\delta\boldsymbol{u}]\!]\cdot\{\boldsymbol{\sigma}\}\,\hat{\boldsymbol{n}}\; d\Gamma + \int_\Gamma \theta[\![\delta\boldsymbol{u}]\!]\cdot[\![\boldsymbol{u}]\!]\,d\Gamma = \int_{\mathcal{B}^+\cup\mathcal{B}^-} \boldsymbol{f}\cdot\delta\boldsymbol{u}\; dV + \int_{\partial\mathcal{B}_t} \boldsymbol{t}\cdot\delta\boldsymbol{u}\; dA \tag{4}$$

Where θ is the penalty parameter. As discussed in equation (3), the jump of displacements $[\![\boldsymbol{u}]\!]$ is equal to zero in the exact solution.

3 NUMERICAL INTEGRATION SCHEME

The dG element includes the information of two adjacent elements, i.e. displacements, strains and stresses. These quantities which appear in the integrands of the weak form (4) are numerically evaluated in two Gauss points on the discontinuity Γ with the means of Gaussian quadrature. Figure 2 illustrates where the Gauss points are located.

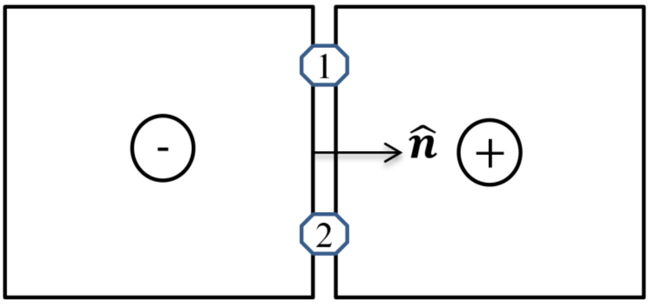

Figure 2. Gauss points 1 and 2 on discontinuity Γ.

The second term and the third term on the left hand side of the weak form (equation (4)), namely the dG term and the penalty term, respectively, are computed in three different ways. First, we evaluate these terms on both Gauss points 1 and 2 as in Figure 2. Then, a fully reduced integration is applied to decrease the Gauss points to one in the middle (Figure 3). Finally, a mixed integration scheme is applied, so that the dG term is evaluated only in the middle of Γ as in Figure 3 and the penalty term on both Gauss points like in Figure 2.

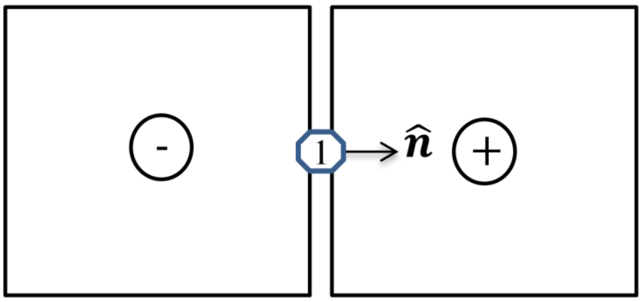

Figure 3. Gauss point 1 in the middle of discontinuity Γ.

4 NUMERICAL EXAMPLE

In this section, we investigate the performance of the dG method in comparison to the standard FEM in a numerical plane strain example with linear elastic isotropic material. The standard quadrilateral finite elements possess four nodes and four Gauss points for numerical integration. In addition, the shape functions are bilinear in both cG and dG elements.

A common benchmark problem [11] of Cook's membrane as shown in Figure 4 is studied. The left side is fixed in both directions and there is an in-plane shear load of 100 N in vertical direction on the right side. The elastic modulus and Poisson's ratio of the material are given by $E = 250\ MPa$ and $\nu = 0.4999$, correspondingly. The vertical displacement of the node P is to be investigated.

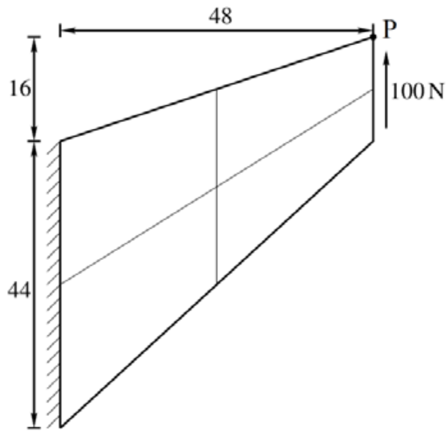

Figure 4. Geometry (in mm), boundary conditions, and loading of Cook's membrane.

Using the Finite Element Analysis Program FEAP to approximate the solution of this problem, we consider two cases of continuous Galerkin (cG) and dG methods. Nevertheless, the dG method was subcategorized into fully reduced and mixed integration schemes.

The results of the simulation depend on two factors, namely the penalty parameter and mesh refinement. Thus, the simulations are carried out for different values of the penalty value θ, varying from 5 to 20,000 and also different mesh sizes. Mesh refinement is done by simultaneous division of the neighboring sides.

Figure 6 and Figure 7 show that for a sufficiently large value of the penalty parameter θ, the results converge with respect to the number of elements. Unsurprisingly, the fully reduced integration scheme converges to wrong solution due to its instability caused by less number of integration points. It is thus more sensitive to the penalty value in comparison to the mixed integration scheme. Taking the θ value greater than 800, the fully reduced integration method converges from 64 element divisions independent of theta value (see Figure 6). Nonetheless, the mixed integration is almost independent of theta, when the number of divisions is higher than 128 (Figure 7).

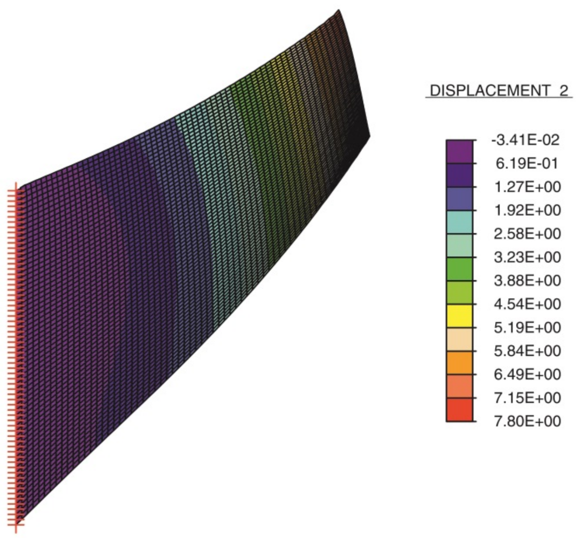

Figure 5. Vertical displacement contour for reduced integration with $\theta = 800$, $n = 64$

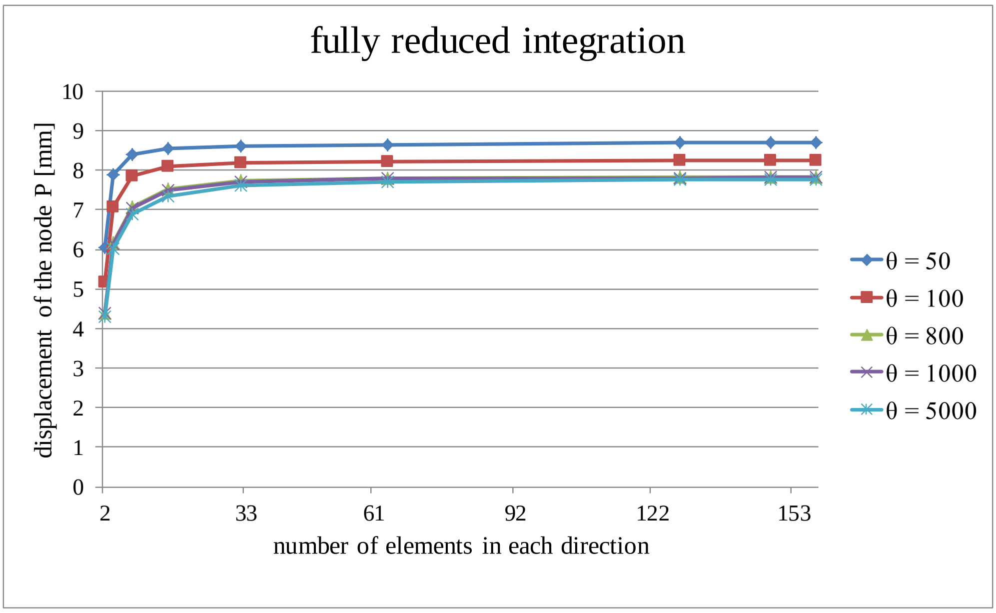

Figure 6. Vertical displacement of the node P for the fully reduced integration scheme

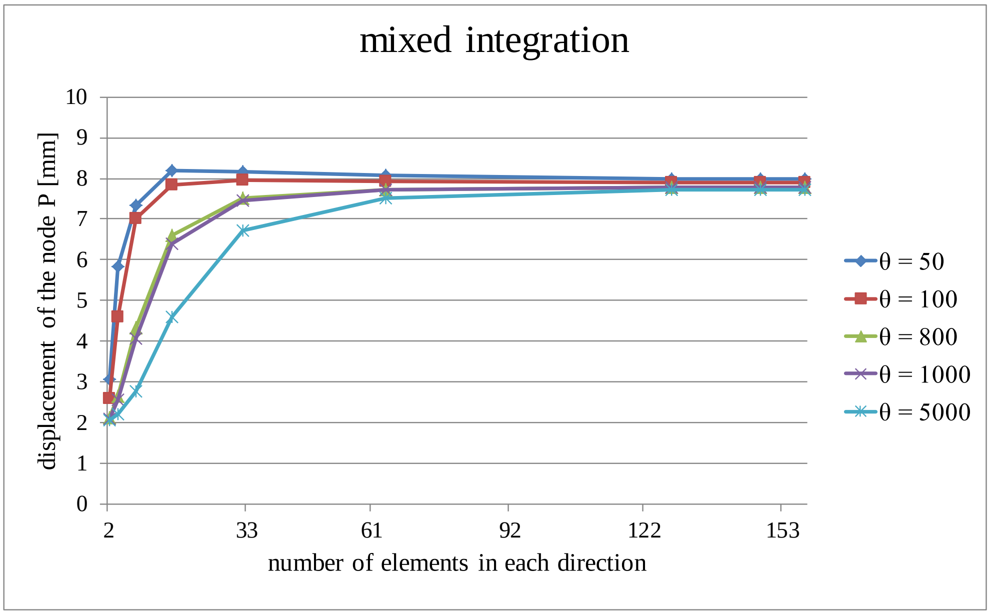

Figure 7. Vertical displacement of the node P for the mixed integration scheme

As it is seen in Figure 8, the dG method converges with much lower number of elements in comparison to the standard continuous Galerkin (cG) method.

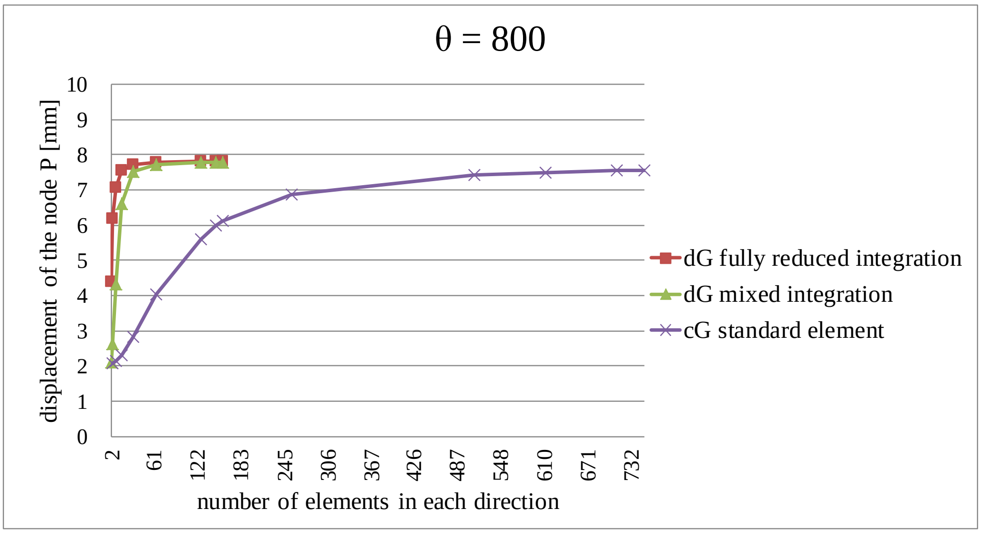

Figure 8. Nodal displacement comparison of cG and dG methods for $\theta = 800$ in vertical direction

5 CONCLUSIONS

In the present work, we utilized the dG method with reduced integration scheme. The results for different cases of the fully reduced and mixed integration methods with respect to IIPG dG terms were examined. Both schemes converged for a specific number of elements and specific value of penalty parameter. Although the dG method converged with a significantly lower number of elements compared to the cG method, one must still consider the major increase of the number of degrees of freedom in dG method, which will bring about a longer calculation time.

6 ACKNOWLEDGMENT

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