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ENERGY-MOMENTUM METHOD FOR NONLINEAR DYNAMIC OF 2D COROTATIONAL BEAMS

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Abstract. *This paper presents an energy-momentum method for nonlinear dynamics of 2D Bernoulli corotational beams. It is shown that the time stepping algorithm conserves energy, linear momentum and angular momentum. To be consistent in the corotational approach, cubic interpolations of Bernoulli element are employed to derive both inertia and elastic terms. The shallow arch strain definition is used to get an element which produce accurate results for less number of elements. To avoid membrane locking, we use a constant and average value of the axial strains. In addition, the energy-momentum method is used to preserve the conserving properties, which is able to maintain the stability and accuracy in a non-dissipative system for a long period. The midpoint velocities of kinematic fields and strains are used to tackle any non-linear form of strain displacement relations. Finally, two examples including large overall displacement are presented to illustrate the stability and efficiency of the proposed algorithms.*

1 INTRODUCTION

The corotational method is an attractive approach to derive non-linear finite beam elements [1, 2, 3, 4, 5, 6, 7, 8] . The idea is to decompose the large motion of the element into rigid body and pure deformational parts through the use of a local system which continuously rotates and translates with the element. The deformational response is captured at the level of the local reference frame, whereas the geometric non-linearity induced by the large rigid-body motion, is incorporated in the transformation matrices relating local and global quantities. The main interest is that the pure deformation part can be assumed as small and can be represented by a linear or a low order non-linear theory.

Regarding corotational dynamic formulations, constant Timoshenko mass matrices [1, 2, 5] are often used to express the dynamic terms. However, such an approach assumes that the in-plane local displacements are zero, which is not accurate. For this reason, Le et al. [3] used the Interpolation Interdependent Element formulation [9], and hence cubic functions, to derive both the elastic and inertia terms. They show that this formulation is more efficient than using constant mass matrices.

Implicit time stepping methods are often used together with non-linear finite elements to study dynamic problems. The Newmark family of algorithms [10] is one of the most commonly employed. However, these methods present instabilities in non-linear analyses [4, 5]. To avoid these instabilities, Geradin and Cardona [11] introduced numerical dissipations (Alpha method [12]) in order to damp the high frequencies. However, by doing that, the energy in the system is not conserved [13, 14].

A different integration scheme, called Energy-Momentum Method, was developed from the standard midpoint rule. Simo and Tarnow [14] were the first authors to use this method that is unconditional stable in non-linear dynamics of three-dimensional elastic bodies. However, their formulation was only valid for quadratic-nonlinearities in the displacement field. In the corotational context, Crisfield and Shi [4] used a mid-point configuration combined to average strains in order to conserve the energy. Galvanetto and Crisfield [5] developed an energy-conserving time-integration procedure for implicit non-linear dynamic analysis of planar beam structures. They used the constant Timoshenko mass matrix for the inertia term.

In this paper, an energy-momentum method for corotational 2D Bernoulli beam elements is proposed. The main advantage of this scheme is that it preserves the total energy and the linear and angular momenta. Besides, this scheme maintains stability and accuracy in long term analyses. Regarding the element, Hermitian shape functions are used to derive both the inertia and elastic terms. A shallow arch strain definition is used for the local formulation. The element formulation is obtained by applying midpoint velocities to both the kinematic quantities and the strains.

2 BEAM KINEMATICS AND STRAIN

2.1 Beam kinematics

The kinematic of the beam and all the notations used in this section are shown in Figure 1. The vector of global displacements is defined by

$$\boldsymbol{q} = \begin{bmatrix} u_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_2 \end{bmatrix}^{\mathrm{T}} \tag{1}$$

The vector of local displacements is defined by

$$\bar{\boldsymbol{q}} = \begin{bmatrix} \bar{u} & \bar{\theta}_1 & \bar{\theta}_2 \end{bmatrix}^{\mathrm{T}} \tag{2}$$

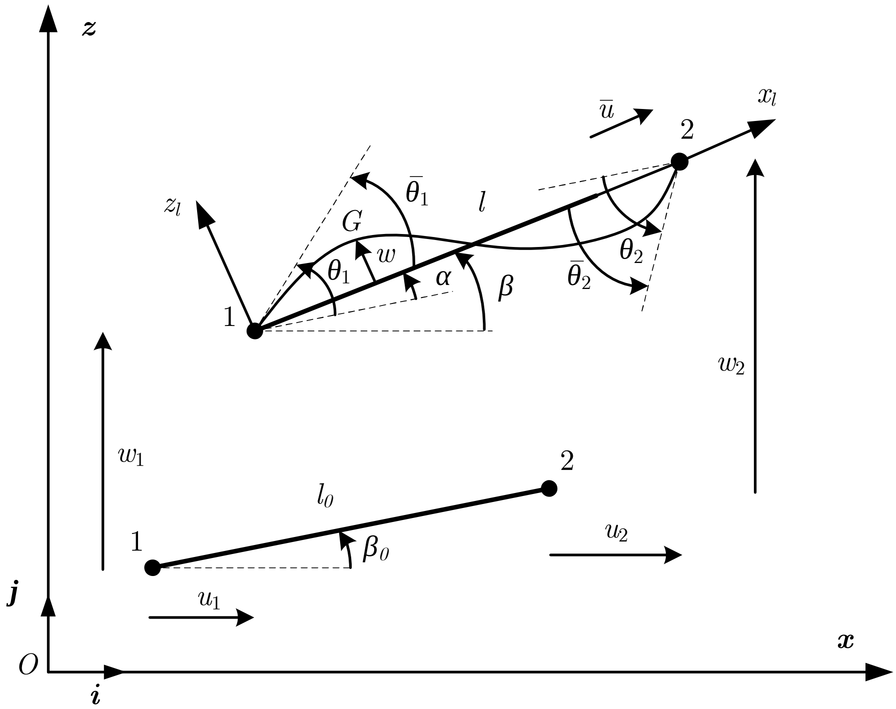

Figure 1: Beam kinematics.

The components of $\bar{\boldsymbol{q}}$ are computed according to

$$\bar{u} = l - l_0 \tag{3}$$
$$\bar{\theta}_1 = \theta_1 - \alpha \tag{4}$$
$$\bar{\theta}_2 = \theta_2 - \alpha \tag{5}$$

where l_0 and l denote the initial and current lengths of the element. The current angle of the local system with respect to the global system is denoted as β and is given by

$$c = \sin\beta = \frac{1}{l}(x_2 + u_2 - x_1 - u_1) \tag{6}$$
$$s = \cos\beta = \frac{1}{l}(z_2 + w_2 - z_1 - w_1) \tag{7}$$

The global position of the centroid G of the cross-section is given by

$$\boldsymbol{OG} = (x_1 + u_1)\,\mathbf{i} + (z_1 + w_1)\,\mathbf{j} + \frac{l_n}{l_0} x\,\boldsymbol{a} + w\,\boldsymbol{b} \tag{8}$$

with

$$\boldsymbol{a} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j} \tag{9}$$
$$\boldsymbol{b} = -\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j} \tag{10}$$

and w is the local transversal displacement of G. By using Eqs.(6) and (7), the components of the global displacements are obtained as

$$u_G = N_1(x_1 + u_1) + N_2(x_2 + u_2) - w\sin\beta \tag{11}$$
$$w_G = N_1(z_1 + w_1) + N_2(z_2 + w_2) + w\cos\beta \tag{12}$$

with

$$N_1 = 1 - \tfrac{x}{l_0} \tag{13}$$
$$N_2 = \tfrac{x}{l_0} \tag{14}$$

The global rotation of the cross-section is given by

$$\theta_G = \vartheta + \alpha \tag{15}$$

where ϑ is the local rotation of the cross-section.

2.2 Strain

The Bernoulli assumption is adopted for the local formulation. Hence, a linear interpolation is taken for the axial displacement u and a cubic one for the vertical displacement w.

The local strain is given by

$$\varepsilon_{11} = \varepsilon - \kappa\, z \tag{16}$$

in which the axial strain ε and the curvature κ are defined by

$$\varepsilon = \frac{1}{l_0}\int_{l_0}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx \tag{17}$$
$$\kappa = \frac{\partial^2 w}{\partial x^2} \tag{18}$$

In Eq.(17), a shallow arch strain definition is taken. The purpose of introducing a low order of geometrical non-linearity in the local formulation is to obtain a more efficient formulation compared to a linear strain definition. The same level of accuracy is obtained with fewer elements. Besides, an average axial strain is taken in order to avoid membrane locking.

3 HAMILTON'S PRINCIPLE AND ENERGY-MOMENTUM METHOD

3.1 Hamilton's principle

Hamilton's principle states that the time integral of the Lagrangian at two specified states between two specified times t_1 and t_2 of a conservative mechanical system is stationary

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0 \tag{19}$$

The integrand $\mathcal{L}$ is called Lagrange function

$$\mathcal{L} = \mathcal{K} - \mathcal{U}_{int} - \mathcal{U}_{ext} \tag{20}$$

$\mathcal{K}$ is the kinetic energy. $\mathcal{U}_{int}$ and $\mathcal{U}_{ext}$ are respectively the internal and the external potential energies. The body is non-conducting linear elastic solid and thermodynamic effects are not included in the system. The kinetic energy is the sum of the translational and rotational kinetic energies:

$$\mathcal{K} = \frac{1}{2}\int_{l_0} \rho A\, \dot{u}_G^2\, dx + \frac{1}{2}\int_{l_0} \rho A\, \dot{w}_G^2\, dx + \frac{1}{2}\int_{l_0} \rho I\, \dot{\theta}_G^2\, dx \tag{21}$$

The elastic potential energy is defined as

$$\mathcal{U}_{int} = \frac{1}{2}\int_{l_0} EA\,\varepsilon^2 dx + \frac{1}{2}\int_{l_0} EI\,\kappa^2 dx \tag{22}$$

The external potential energy is defined as

$$\mathcal{U}_{ext} = -\int_{l_0} p_u \cdot u_G\,dx - \int_{l_0} p_w \cdot w_G\,dx - \int_{l_0} p_\theta \cdot \theta_G\,dx - \sum_{i=1}^{6} P_i \cdot q_i \tag{23}$$

E is Young's modulus of the material, A is the area of the cross-section, I is the inertia moment of the cross-section, p_u and p_w are the distributed horizontal and the vertical loads, p_θ is the distributed external moment, P_i is the vector of concentrated forces and moments at the nodes.

By introducing Eqs.(21) to (23), the variation of Eq.(19) can be written as

$$\begin{aligned}
& \int_{t_1}^{t_2} \left(\int_{l_0} \rho A\,\dot{u}_G \cdot \delta\dot{u}_G\,dx + \int_{l_0} \rho A\,\dot{w}_G \cdot \delta\dot{w}_G\,dx + \int_{l_0} \rho I\,\dot{\theta}_G \cdot \delta\dot{\theta}_G\,dx - \int_{l_0} EA\,\varepsilon \cdot \delta\varepsilon\,dx \right) \\
- & \int_{t_1}^{t_2} \left(\int_{l_0} EI\,\kappa \cdot \delta\kappa\,dx - \int_{l_0} p_u \cdot \delta u_G\,dx - \int_{l_0} p_w \cdot \delta w_G\,dx - \int_{l_0} p_\theta \cdot \delta\theta_G\,dx \right) \\
+ & \int_{t_1}^{t_2} \sum_{i=1}^{6} P_i \cdot \delta q_i = 0
\end{aligned} \tag{24}$$

By using part integration for the first three terms of (24), the previous equation can be reformulated

$$\begin{aligned}
& \int_{t_1}^{t_2} \left(\int_{l_0} \rho A\,\ddot{u}_G \cdot \delta u_G\,dx + \int_{l_0} \rho A\,\ddot{w}_G \cdot \delta w_G\,dx + \int_{l_0} \rho I\,\ddot{\theta}_G \cdot \delta\theta_G\,dx \right) \\
+ & \int_{t_1}^{t_2} \left(\int_{l_0} EA\,\varepsilon \cdot \delta\varepsilon\,dx + \int_{l_0} EI\,\kappa \cdot \delta\kappa\,dx - \int_{l_0} p_u \cdot \delta u_G\,dx - \int_{l_0} p_w \cdot \delta w_G\,dx \right) \\
+ & \int_{t_1}^{t_2} \left(-\int_{l_0} p_\theta \cdot \delta\theta_G\,dx - \sum_{i=1}^{6} P_i \cdot \delta q_i \right) = 0
\end{aligned} \tag{25}$$

3.2 Energy-momentum integration scheme

The classical midpoint time integration scheme is defined by the following equations:

$$\boldsymbol{q}_{n+\frac{1}{2}} = \frac{\boldsymbol{q}_{n+1} + \boldsymbol{q}_n}{2} = \boldsymbol{q}_n + \frac{1}{2}\Delta\boldsymbol{q} \tag{26}$$

$$\dot{\boldsymbol{q}}_{n+\frac{1}{2}} = \frac{\dot{\boldsymbol{q}}_{n+1} + \dot{\boldsymbol{q}}_n}{2} = \frac{\boldsymbol{q}_{n+1} - \boldsymbol{q}_n}{\Delta t} = \frac{\Delta\boldsymbol{q}}{\Delta t} \tag{27}$$

$$\ddot{\boldsymbol{q}}_{n+\frac{1}{2}} = \frac{\ddot{\boldsymbol{q}}_{n+1} + \ddot{\boldsymbol{q}}_n}{2} = \frac{\dot{\boldsymbol{q}}_{n+1} - \dot{\boldsymbol{q}}_n}{\Delta t} = \frac{2}{\Delta t^2}\Delta\boldsymbol{q} - \frac{2}{\Delta t}\dot{\boldsymbol{q}}_n \tag{28}$$

By extension of the classical midpoint rule, the average midpoint strains are developed in the context of energy-momentum method. This idea has been introduced in [16, 17]. The midpoint velocities are applied to both the kinematic fields and the strains because the nonlinear terms arise from both fields. This gives:

$$\int_{t_n}^{t_{n+1}} f(t)dt = f(t_{n+\frac{1}{2}})\Delta t = f_{n+\frac{1}{2}}\Delta t \tag{29}$$

$$f_{n+\frac{1}{2}} = f_n + \frac{\Delta t}{2}\dot{f}_{n+\frac{1}{2}} \tag{30}$$

where the function f can represent both the kinematic (u_G, w_G, θ_G) and deformational quantities (ε, κ).

The application of the midpoint rule (29) to the Hamilton's principle (25) gives

$$
\begin{aligned}
\Delta t \quad & \Big(\int_{l_0} \rho A\, \ddot{u}_{G,n+\frac{1}{2}} \cdot \delta u_{G,n+\frac{1}{2}}\, dx + \int_{l_0} \rho A\, \ddot{w}_{G,n+\frac{1}{2}} \cdot \delta w_{G,n+\frac{1}{2}}\, dx + \int_{l_0} \rho I\, \ddot{\theta}_{G,n+\frac{1}{2}} \cdot \delta \theta_{G,n+\frac{1}{2}}\, dx \\
+ \quad & \int_{l_0} EA\, \varepsilon_{n+\frac{1}{2}} \cdot \delta \varepsilon_{n+\frac{1}{2}}\, dx + \int_{l_0} EI\, \kappa_{n+\frac{1}{2}} \cdot \delta \kappa_{n+\frac{1}{2}}\, dx - \int_{l_0} p_{u,n+\frac{1}{2}} \cdot \delta u_{G,n+\frac{1}{2}}\, dx \\
- \quad & \int_{l_0} p_{w,n+\frac{1}{2}} \cdot \delta w_{G,n+\frac{1}{2}}\, dx - \int_{l_0} p_{\theta,n+\frac{1}{2}} \cdot \delta \theta_{G,n+\frac{1}{2}}\, dx - \sum_{i=1}^{6} P_{i,n+\frac{1}{2}} \cdot \delta q_{i,n+\frac{1}{2}} \Big) = 0
\end{aligned}
\tag{31}
$$

As the variation $\delta \boldsymbol{q}$ is arbitrary, the dynamic equilibrium at time $n + \frac{1}{2}$ is obtained from the previous equation

$$
\begin{aligned}
& \int_{l_0} \rho A\, \ddot{u}_{G,n+\frac{1}{2}} \frac{\partial u_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx + \int_{l_0} \rho A\, \ddot{w}_{G,n+\frac{1}{2}} \frac{\partial w_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx + \int_{l_0} \rho I\, \ddot{\theta}_{G,n+\frac{1}{2}} \frac{\partial \theta_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx \\
+ \quad & \int_{l_0} EA\, \varepsilon_{n+\frac{1}{2}} \frac{\partial \varepsilon_{n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx + \int_{l_0} EI\, \kappa_{n+\frac{1}{2}} \frac{\partial \kappa_{n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx - \int_{l_0} p_{u,n+\frac{1}{2}} \frac{\partial u_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx \\
- \quad & \int_{l_0} p_{w,n+\frac{1}{2}} \frac{\partial w_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx - \int_{l_0} p_{\theta,n+\frac{1}{2}} \frac{\partial \theta_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}} dx - P_{i,n+\frac{1}{2}} = 0
\end{aligned}
\tag{32}
$$

4 NUMERICAL EXAMPLES

Two numerical applications are presented in this section. The first purpose of these examples is to verify numerically that the proposed energy-momentum algorithm conserves the total energy of the system and remains stable even if a very large number of time steps are applied. The second purpose is to show that in the absence of applied external loads, the proposed algorithm conserves the linear and angular momenta.

4.1 Cantilever beam

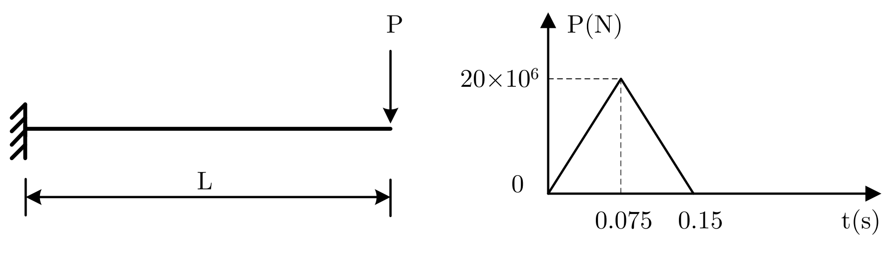

Figure 2: Geometry and load history.

The first example, see Figure 2, is a cantilever beam loaded by a triangular force at its end. The parameters of the problem are:

$$
\begin{aligned}
L &= 3\ \textit{m}, A = 1000\ \textit{cm}^2, I = 8330\ \textit{cm}^4 \\
E &= 200\ \textit{GPa}, \rho = 48831\ \textit{kg/m}^3 \\
\Delta t &= 10^{-3}\ \textit{s}, \textit{Number of elements} = 4
\end{aligned}
$$

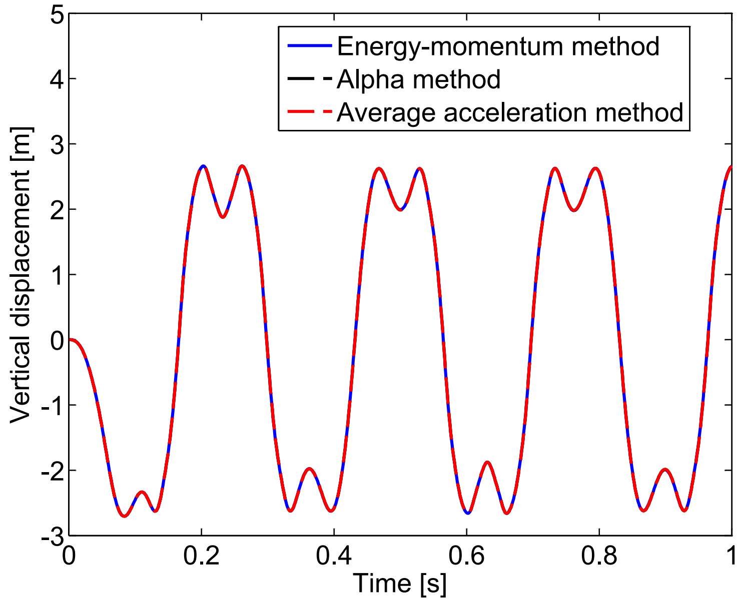

Figure 3: Comparison the displacements at the end of the tip.

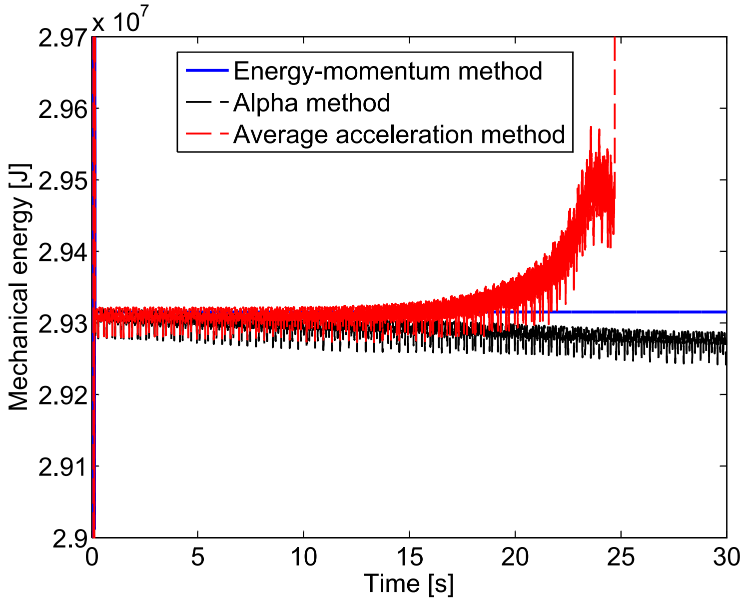

Figure 4: Comparison of mechanical energy from 0s to 30s.

The results obtained with the present energy-momentum formulation are compared to the ones obtained with the corotational formulation proposed by Le et al. [3]. In this previous formulation, the alpha method is used to solve the equations of motion. Two cases are considered here, $\alpha = 0$, which corresponds to the classical average acceleration method and $\alpha = -0.01$ which gives a small numerical damping that limits the influence of higher modes on the response.

The results presented in Figure 3 show that at the beginning the three approaches give exactly the same results. The results in Figures 4 and 5 show clearly that with the average acceleration

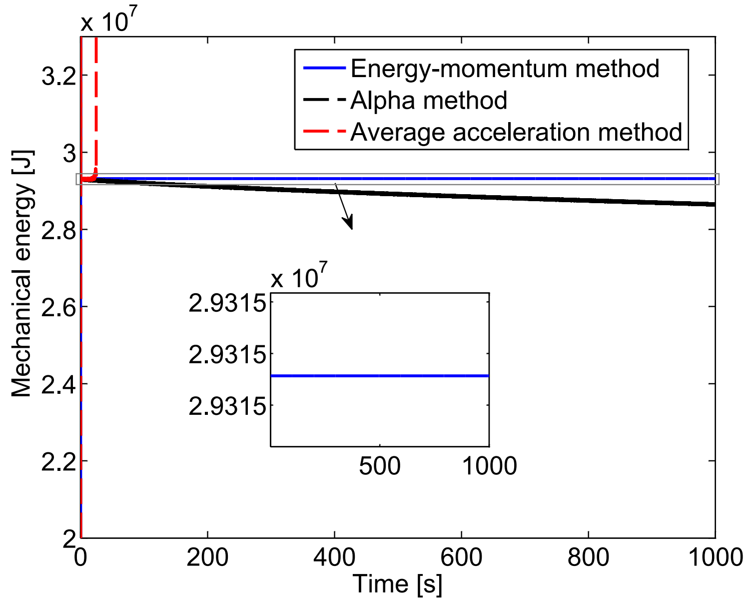

Figure 5: Comparison of mechanical energy from 0s to 1000s.

method, the mechanical energy (sum of kinetic energy and internal energy) blows up after about $24s$ and the solution diverges. With the alpha method, the solution does not diverge, but as expected, there is a loss of the mechanical energy due to the numerical damping. However, with the present energy-momentum approach, the mechanical energy is constant and the solution remains stable even if a very large number of steps (one million) is applied.

4.2 Free fly beam

The second example is a free flying beam, see Figure 6. The parameters of the problems are

$$\begin{aligned} L &= 3\,m, A = 200\ cm^2, I = 66.67\ cm^4 \\ E &= 200\ GPa, \rho = 48831\ kg/m^3 \\ \Delta t &= 10^{-4}\ s, Number\ of\ elements = 4 \end{aligned}$$

The interest of this problem is that after 0.4s, no forces and moments are applied to the beam. Consequently, this problem is suitable to study the conservation of the linear and angular momenta. Figures 7, 8 and 9 show the conservation of energy and momenta for one million time steps. The relative error is about 10^{-8} for the energy. We present here the two momenta on two different figures 8 and 9 with appropriate scale. The figures show that all the moments are constant.

5 CONCLUSION

This paper has presented an energy-momentum dynamic integration scheme in the context of corotational 2D beam elements. The main idea is to use the classical midpoint rules for both the kinematic and strain quantities. The advantage of the propose algorithm is that it conserves the total energy of the system and remains stable and accurate even if a very large number of time steps are applied. Besides, in the absence of applied external loads, the linear and angular momenta are constant. These characteristics have been proved numerically by using two numerical applications.

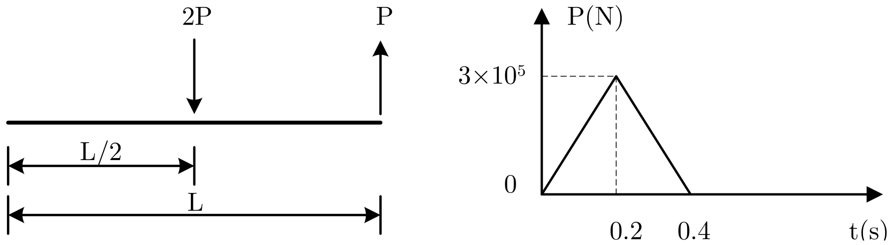

Figure 6: Geometry and load history of free fly beam.

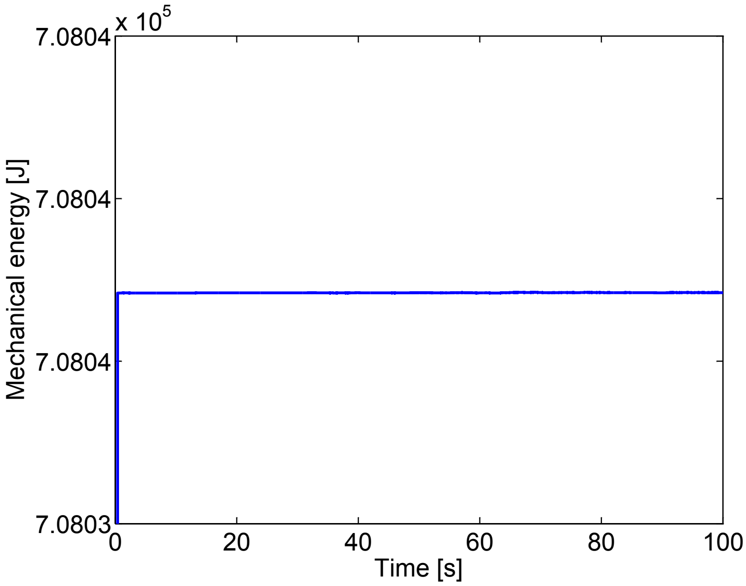

Figure 7: Mechanical energy.

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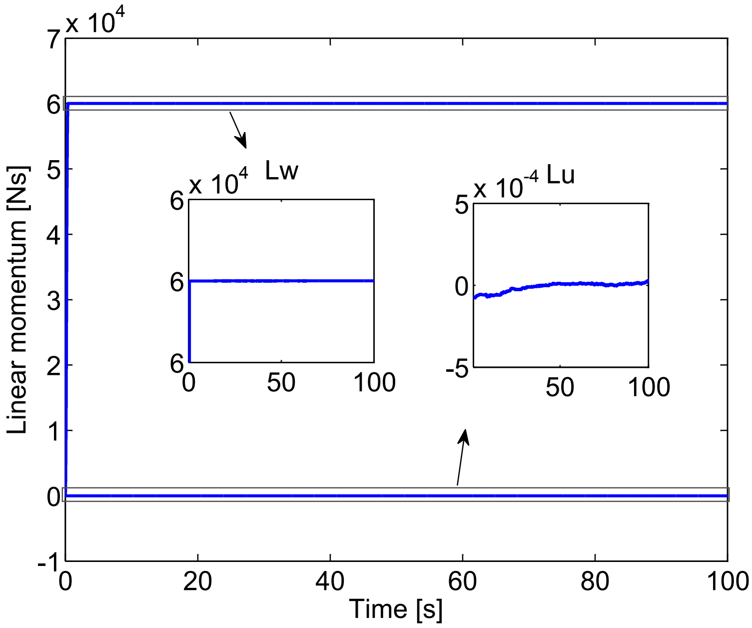

Figure 8: Linear momentum.

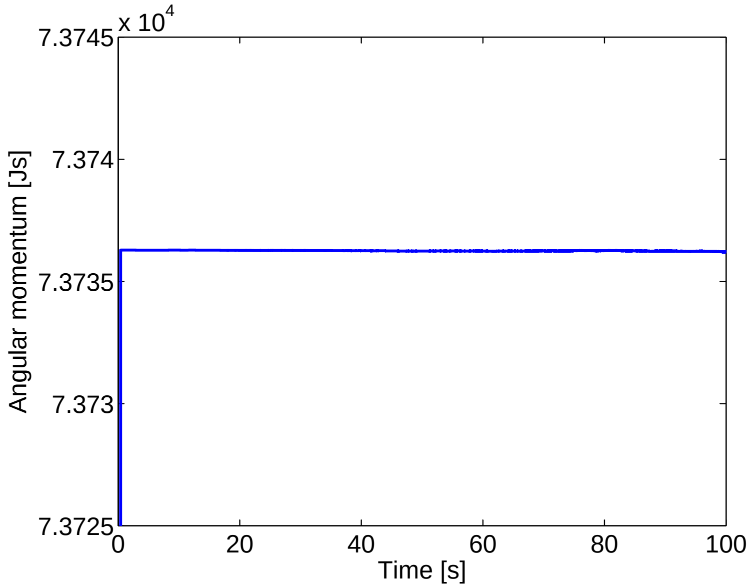

Figure 9: Linear momentum.

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