

## A MULTIGRID FORMULATION FOR FINITE DIFFERENCE METHODS ON SUMMATION–BY–PARTS FORM: AN INITIAL INVESTIGATION

Andrea A. Ruggiu<sup>1</sup>, Per Weinerfelt<sup>1,2</sup>, Tomas Lundquist<sup>1</sup> and Jan Nordström<sup>1</sup>

<sup>1</sup>Department of Mathematics, Computational Mathematics, Linköping University,  
SE-581 83 Linköping, Sweden  
e-mail: {andrea.ruggiu,per.weinerfelt,tomas.lundquist,jan.nordstrom}@liu.se

<sup>2</sup>Saab Aerospace,  
SE-581 88 Linköping, Sweden  
e-mail: per.weinerfelt@saab.se

**Keywords:** high order finite difference methods, summation–by–parts, multigrid, restriction and prolongation operators, improved convergence.

**Abstract.** *Several multigrid iteration schemes for high order finite difference methods are studied by comparing the effect of different interpolation operators. The usual choice of prolongation and restriction operators based on linear interpolation in combination with the Galerkin condition leads to coarse grid operators which are less accurate than their fine grid counterparts.*

*Moreover, these operators do not mimic the integration–by–parts property possessed by the original fine grid summation–by–part schemes and hence are intuitively less stable. In this paper, an alternative class of interpolation operators is considered to overcome these issues and improve the stability of the overall multigrid iteration scheme. As a pleasant side effect we find that also the efficiency of the iteration scheme is improved.*

## 1 INTRODUCTION

The multigrid method is a convergence acceleration technique based on a hierarchy of grids [1, 2]. The solution of a discrete problem on a fine grid is computed by solving a system of equations on the coarsest grid level. This technique is applied to problems in various branches of applied mathematics and engineering, such as the Helmholtz equation [3], magnetohydrodynamics [4] and computational fluid dynamics [5].

One of the main issues associated with this method is the construction of restriction and prolongation operators which transfer the information between grids. In this work we make use of a general relation between prolongation and restriction operators which was originally proposed in a different context [6]. These interpolation operators are consistent at the boundaries and guarantee that the accuracy of the original scheme is retained at the interior nodes on each grid level.

By using this result, we combine the multigrid method with an energy stable discretization for high order finite difference methods, based on the Summation-by-Parts (SBP) and Simultaneous-Approximation-Term (SAT) approach described in [7]. The improved stability of the new multigrid method is shown through numerical simulations of a hyperbolic problem in one dimension.

The rest of this paper is organized as follows: in Section 2 the main features of the two-level multigrid algorithm are presented. Section 3 introduces the SBP operators and the SAT penalty terms for high order finite difference discretizations. Section 4 deals with the construction of multigrid algorithms for SBP-SAT discretizations using the new class of interpolation operators. In Section 5 the new algorithm is compared to a multigrid scheme with conventional prolongation and restriction operators. Conclusions are drawn in Section 6.

## 2 THE MULTIGRID ALGORITHM

In this Section the multigrid approach is outlined by means of a two-level algorithm [1, 2, 8]. Let us consider the following steady-state problem

$$\begin{aligned} Lu &= f, & \text{in } \Omega, \\ Hu &= g, & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where  $L$  is a differential operator,  $H$  is a boundary operator,  $f$  and  $g$  are given functions, and  $\Omega$  is a domain. We assume that the boundary conditions are assigned in a way such that (1) is well-posed. The multigrid algorithm for solving (1) can be written in four steps:

1. Discretization;
2. Error smoothing;
3. Coarse-grid correction;
4. Fine-grid update.

### 2.1 Discretization

Construct a discretization for (1) on a grid  $\Omega_1 \subset \Omega \cup \partial\Omega$  called *fine grid*. The resulting problem has the form

$$L_1 \mathbf{u} = \mathbf{f}, \tag{2}$$

where  $L_1$  is a discretization of the operator  $L$  in (1) which also includes the boundary conditions defined by  $H$ . The vector  $\mathbf{f}$  is a grid function which approximates  $f$  on the nodes of  $\Omega_1$ , possibly augmented with boundary data, and  $\mathbf{u}$  is an approximate solution to (1). Typically  $\Omega_1$  has many nodes, and therefore it may be expensive to solve (2) directly.

The discrete operator  $L_1$  is assumed to be invertible, since its continuous counterpart  $L$  in (1), with the boundary conditions defined by  $H$ , leads to a well-posed problem. Even though we do not want to compute the inverse of  $L_1$ , the solution to (2) will be indicated by  $L_1^{-1}\mathbf{f}$  in the rest of the paper.

## 2.2 Error smoothing

Define an error smoothing procedure. As an example, we may consider to march toward the solution to (2) from an initial guess  $\mathbf{u}^{(0)}$  by solving

$$\begin{aligned} \left(\frac{d}{dt} + L_1\right) \mathbf{w}(t) &= \mathbf{f}, & t > 0, \\ \mathbf{w}(0) &= \mathbf{u}^{(0)}, \end{aligned} \quad (3)$$

for a given finite time  $\Delta t > 0$  called *smoothing step*. The solution to (3) is

$$\mathbf{w}(\Delta t) = e^{-L_1\Delta t}\mathbf{u}^{(0)} + (I - e^{-L_1\Delta t})L_1^{-1}\mathbf{f}, \quad (4)$$

where  $I$  indicates the identity matrix on  $\Omega_1$ . Note that this technique yields a smoothing effect on the grid function  $\mathbf{u}^{(0)}$  if all the eigenvalues of  $L_1$  have positive real part. In particular,  $\mathbf{w}(\Delta t) \rightarrow \mathbf{u} = L_1^{-1}\mathbf{f}$  as  $\Delta t \rightarrow +\infty$ .

By similarity, we may define a general smoothing technique for (2) as

$$\begin{aligned} \mathbf{v}^k &= S\mathbf{v}^{k-1} + (I - S)L_1^{-1}\mathbf{f}, & k = 1, \dots, \nu, \\ \mathbf{v}^0 &= \mathbf{u}^{(0)}, \end{aligned} \quad (5)$$

where the exponential smoother  $S_{exp} = e^{-L_1\Delta t}$  yields a procedure based on time-marching as in (4). After  $\nu$  steps, the iterative method (5) provides the grid function

$$\mathbf{v} = S^\nu\mathbf{u}^{(0)} + (I - S^\nu)L_1^{-1}\mathbf{f}. \quad (6)$$

This step of the algorithm is necessary since the high frequency modes on  $\Omega_1$  may be unresolved on coarser grids (see Figure 1).

## 2.3 Coarse-grid correction

Consider  $\Omega_2 \subsetneq \Omega_1$ , called the *coarse grid*, and solve a correction problem on  $\Omega_2$ :

$$L_2\mathbf{d} = I_r(\mathbf{f} - L_1\mathbf{v}) \quad (7)$$

which is obtained from (2) and (4) by making use of

- a restriction operator  $I_r : \Omega_1 \rightarrow \Omega_2$ ,
- a coarse-grid operator  $L_2 : \Omega_2 \rightarrow \Omega_2$ .

The coarse-grid operator can be built by using the Galerkin Condition [2]

$$L_2 = I_r L_1 I_p, \quad (8)$$

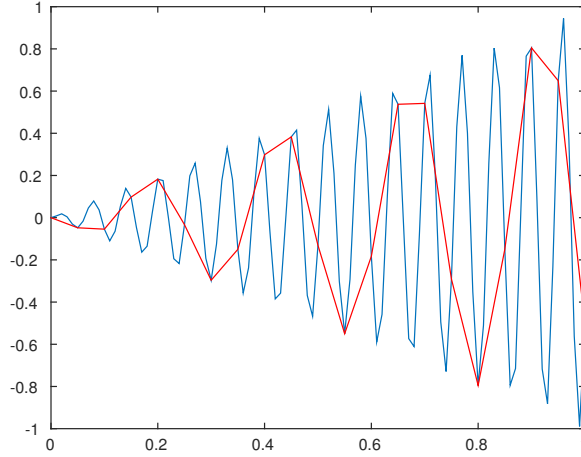


Figure 1: The function  $x \sin(100x)$ ,  $x \in [0, 1]$ , is represented on a fine (blue line) and on a coarse grid (red line). The fine grid is given by 101 equidistant points, while the coarse grid has got 21 nodes. The restricted grid function has a lower frequency than the original one.

where  $I_p : \Omega_2 \rightarrow \Omega_1$  is a prolongation operator.

A common choice for  $I_p$  in the one dimensional case is based on linear interpolation. Assuming for example that  $\Omega_1 = \{x_j : x_j = jh, j = 0, \dots, N\}$  with  $h = 1/N$ , and that  $\Omega_2$  consists of the odd nodes of  $\Omega_1$ , we may write

$$(I_p \mathbf{v})_m = \begin{cases} v_j, & m = 2j, \\ \frac{1}{2}(v_j + v_{j+1}), & m = 2j + 1, \end{cases} \quad j = 0, \dots, N/2, \quad (9)$$

while the restriction operator is usually given in terms of the prolongation operator

$$I_r = \frac{1}{2} I_p^T. \quad (10)$$

## 2.4 Fine-grid update and multigrid iteration scheme

Update the fine grid solution  $\mathbf{v}$  with an estimate of the correction  $\mathbf{d}$  in (7), obtained through prolongation with  $I_p$

$$\mathbf{u}^{(1)} = \mathbf{v} + I_p \mathbf{d}. \quad (11)$$

This relation, together with (6) and (7), provides an iterative method for solving (1):

$$\mathbf{u}^{(n+1)} = M \mathbf{u}^{(n)} + N \mathbf{f}, \quad (12)$$

where

$$M = CS^\nu, \quad C = I - I_p L_2^{-1} I_r L_1 \quad \text{and} \quad N = (I - M) L_1^{-1}. \quad (13)$$

We will refer to  $M$  as the multigrid iteration matrix and to  $C$  as the coarse-grid correction operator.

The role of  $M$  in the convergence of the iterative method (12) is central: by considering the errors  $\mathbf{e}^{(n)} = \mathbf{u}^{(n)} - L^{-1} \mathbf{f}$ , we get

$$\mathbf{e}^{(n+1)} = M \mathbf{e}^{(n)}, \quad (14)$$

which leads to convergence if the spectral radius of  $M$  is less than 1.

## 2.5 The coarse–grid correction operator

In Section 2.3 we claimed that the Galerkin condition (8) may be used to build a coarse–grid operator  $L_2$ . This choice leads to some interesting theoretical properties about the coarse–grid correction operator  $C$  in (12), (13). To start with, we prove

**Lemma 2.1.** *If the Galerkin condition (8) holds, then the coarse–grid correction operator  $C$  in (12), (13) is idempotent, i.e.  $C^k = C$  for every  $k \in \mathbb{N}$ .*

*Proof.* By multiplying  $C$  by itself we get

$$C^2 = (I - I_p L_2^{-1} I_r L_1) (I - I_p L_2^{-1} I_r L_1) = I - 2I_p L_2^{-1} I_r L_1 + I_p L_2^{-1} I_r L_1 I_p L_2^{-1} I_r L_1 = C,$$

since the Galerkin condition  $L_2 = I_r L_1 I_p$  is fulfilled.  $\square$

The idempotency of  $C$  implies that its eigenvalues are given by zeros and ones [9]. Moreover, it is possible to show

**Corollary 2.2.** *If the Galerkin condition (8) holds, the image of the prolongation operator  $I_p$  is contained in the nullspace of  $C$ .*

*Proof.* Consider a vector  $\mathbf{y} \in \text{Im}(I_p)$ , i.e.  $\mathbf{y} = I_p \mathbf{z}$ . When the coarse–grid correction operator acts on this vector, we find

$$C\mathbf{y} = (I - I_p L_2^{-1} I_r L_1) I_p \mathbf{z} = I_p \mathbf{z} - I_p L_2^{-1} I_r L_1 I_p \mathbf{z} = \mathbf{0},$$

since the Galerkin condition holds.  $\square$

The main consequence of Corollary 2.2 is that  $C$  cancels all the grid functions which can be represented through the prolongation operator  $I_p$ . This implies that  $C$  deals with smooth errors while the smoother  $S$  damps the remaining high frequency modes.

## 3 THE SBP–SAT DISCRETIZATION

In this Section we introduce the Summation–by–Parts operators (SBP) and the Simultaneous–Approximation–Term (SAT) technique. The first ones mimic integration by parts, whereas the second one introduces penalty–like terms enforcing the boundary conditions weakly [7].

**Definition 3.1.** *We say that  $D = P^{-1}Q$  is a  $(p, q)$ –accurate first derivative SBP operator, if  $Q + Q^T = B = \text{diag}(-1, 0, \dots, 0, 1)$ ,  $P$  is a symmetric positive definite matrix, and the associated truncation errors are  $\mathcal{O}(h^p)$  in the interior and  $\mathcal{O}(h^q)$  at the boundaries.*

SBP operators based on diagonal norms  $P$  are available for even orders  $p = 2q$  in the interior, while the boundary closure is  $q$ th order accurate. Even though this is not the only possible choice, we will only consider these  $(2q, q)$ –accurate operators with diagonal  $P$ . For further details on the construction of SBP operators for the first derivative with  $q \leq 4$ , see [10].

The SBP finite difference operators together with a strong treatment of the boundary conditions only admits stability proofs for very simple problems. This was shown in [11], where SAT technique was proposed to complement the SBP schemes. By discretizing a well–posed Initial–Boundary–Value–Problem (IBVP) with both SBP operators and SAT penalty terms (the SBP–SAT approach), it is possible to prove that the corresponding semi–discrete problem is stable. As an example, we consider the advection equation in one dimension.

### 3.1 The advection problem

Consider

$$\begin{aligned} u_t + u_x &= 0, & 0 < x < 1, & t > 0, \\ u(x, 0) &= f(x), & 0 < x < 1, & \\ u(0, t) &= g(t), & t > 0, & \end{aligned} \quad (15)$$

where both  $f$  and  $g$  are known data. The problem (15) is well-posed, since the analytical solution is a traveling wave, and the boundary condition is imposed at the inflow.

Let us consider an  $(N + 1)$ -point uniform grid on  $[0, 1]$ , given by  $x_j = jh$  for  $j = 0, \dots, N$ , where  $h = 1/N$ . We introduce the grid function  $f_j = f(x_j)$  and to each grid point we also associate the approximate solution  $u_j$ . By applying the SBP-SAT discretization in space to (15), we get

$$\begin{aligned} \mathbf{u}_t + P^{-1}Q\mathbf{u} &= P^{-1}\sigma(u_0 - g)\mathbf{e}_0, & t > 0, \\ \mathbf{u}(0) &= \mathbf{f}, \end{aligned} \quad (16)$$

where  $\mathbf{u} = (u_0, \dots, u_N)$ ,  $\mathbf{f} = (f_0, \dots, f_N)$ ,  $\sigma \in \mathbb{R}$  is a penalty parameter which can be tuned for stability, and  $\mathbf{e}_0 = [1, 0, \dots, 0]^T \in \mathbb{R}^{N+1}$ . By defining the matrix  $\tilde{Q}_\sigma = Q - \sigma\mathbf{e}_0\mathbf{e}_0^T$  and the vector  $\mathbf{F} = -P^{-1}\sigma g\mathbf{e}_0$  we can rewrite (15) in a more compact form

$$\begin{aligned} \mathbf{u}_t + P^{-1}\tilde{Q}_\sigma\mathbf{u} &= \mathbf{F}, & t > 0, \\ \mathbf{u}(0) &= \mathbf{f}, \end{aligned} \quad (17)$$

where  $\tilde{Q}_\sigma + \tilde{Q}_\sigma^T = \text{diag}(- (1 + 2\sigma), 0, \dots, 0, 1)$ .

We want to find  $\sigma$  such that the problem (17) is strongly stable, i.e. such that

$$\|\mathbf{u}(\cdot, t)\|^2 \leq K(t) \left( \|\mathbf{f}\|^2 + \max_{\tau \in [0, t]} \|g(\tau)\|^2 \right) \quad (18)$$

in a suitable norm, with  $K(t)$  bounded for any finite  $t$  and independent of the data (for further details, see [7]). By multiplying the equation (17) by  $\mathbf{u}^T P$  and adding the transpose we find

$$\frac{d}{dt} \|\mathbf{u}\|_P^2 = -u_N^2 + (1 + 2\sigma)u_0^2 - 2\sigma g u_0$$

where  $\|\mathbf{u}\|_P = \sqrt{\mathbf{u}^T P \mathbf{u}}$ . For  $g = 0$  the method is stable for  $\sigma \leq -\frac{1}{2}$ . If  $g \neq 0$  and assuming that  $\sigma \neq -\frac{1}{2}$ , we get

$$\frac{d}{dt} \|\mathbf{u}\|_P^2 = -\frac{\sigma^2}{1 + 2\sigma} g^2 - u_N^2 + \frac{[(1 + 2\sigma)u_0 - \sigma g]^2}{1 + 2\sigma},$$

which lead to an estimate of the form (18). Therefore, the semi-discrete problem (16) is strongly stable if  $\sigma < -\frac{1}{2}$ .

## 4 THE SBP-SAT-MG SCHEME

In the previous Section we have shown that operators which mimic integration by parts leads to stable semi-discrete problems. We conjecture that preserving this property on both fine and coarse levels should lead to an improved multigrid method.

Let us consider the following steady model problem

$$\begin{aligned} u_x &= f, & 0 < x < 1, \\ u(0) &= g. \end{aligned} \quad (19)$$



**Proposition 4.3.** *Consider the  $2q$ th-order accurate SBP-preserving interpolation operators  $I_p$  and  $I_r$ , with  $q \geq 1$ . If  $D$  is a  $(2q, q)$ -first derivative SBP-operator in the fine grid, then  $D_2 = I_r D I_p$  is a first derivative SBP-operator in the coarse grid. Moreover, it is  $2q$ th-order accurate in the interior nodes, while the accuracy on the boundary nodes is at least  $q - 1$ .*

The proof will be given in the full paper.

**Remark 4.4.** *For  $q = 1$  the coarse grid operator  $D_2$  has the same structure as  $D$  and is first-order accurate at the boundaries.*

The operators for higher orders will be provided in the full paper.

## 5 NUMERICAL EXPERIMENTS

In this Section we perform numerical computations to test the properties of the new multigrid iteration scheme. Consider (19) as model problem with a manufactured solution given by

$$v(x) = e^{-x} (\cos(10\pi x) + \cos(2\pi x)), \quad 0 < x < 1.$$

To discretize (19) we use the formulation (20) on an  $(N+1)$ -point uniform grid on  $[0, 1]$ . The tests are carried out using a single-step smoother based on a modified Runge-Kutta scheme which was proposed in [12] for first order systems:

$$S = I - \Delta t L_1 + 0.6 \Delta t^2 L_1^2 - 0.36 \Delta t^3 L_1^3. \quad (23)$$

In the following tests, we consider that the multigrid iteration has reached convergence if  $\|\mathbf{u}^{n+1} - \mathbf{u}^n\| < \varepsilon$ , where  $\varepsilon = \text{tolerance}$ .

### 5.1 IMPROVED CONVERGENCE

Consider a  $(2, 1)$ -order accurate discretization on a grid with  $N = 500$  and a smoothing step  $\Delta t = 5 \cdot 10^{-4}$ . Figure 2 shows the spectral radius and the distribution of the eigenvalues of the multigrid iteration matrix  $M$  in (13) by varying the interpolation operators. In particular, the SBP-preserving interpolation operators of 2nd order are tested against the classical choices (9) and (10). Note that even if in this case the prolongation operator is the same (see (22)), a minor change in the restriction operator affects the distribution of the eigenvalues of  $M$ .

To test the convergence properties, we have used both the tolerance  $10^{-10}$  and  $10^{-3}$ . The results, shown in Table 1, suggest that the new class of operator is clearly superior.

|                                     | SBP-preserving | Conventional |
|-------------------------------------|----------------|--------------|
| Number of iterations ( $10^{-10}$ ) | 2372           | $> 10^6$     |
| Number of iterations ( $10^{-3}$ )  | 1470           | 1541         |

Table 1: The number of iterations to convergence for the multigrid scheme with tolerance  $10^{-10}$  and  $10^{-3}$ .

Another test with tolerance  $\varepsilon = 10^{-10}$  and an increased smoothing step  $\Delta t = 6.3 \cdot 10^{-4}$  makes the spectral radius for the SBP-preserving choice decrease, while the classical linear prolongation and restriction lead to instability. As a side effect, the number of iterations to convergence for the new interpolation operators drops to 1873. In Figure 3 the distributions of the eigenvalues are shown for this case. Note that the current smoothing step is close to the



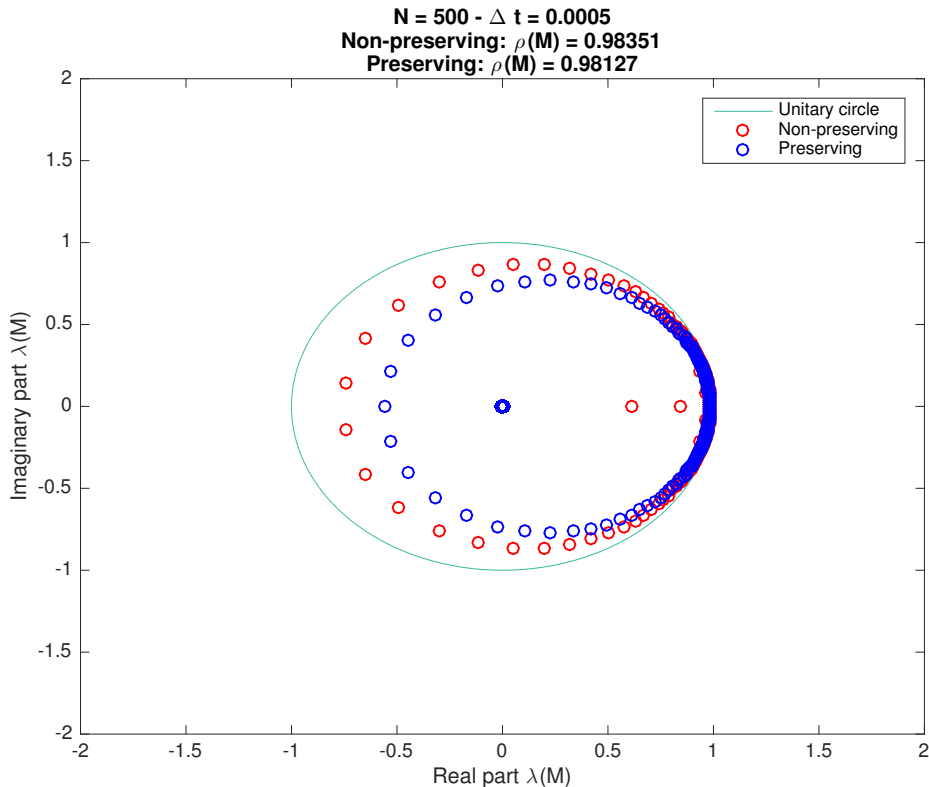


Figure 2: Eigenvalues of the multigrid iteration matrix for different choices of interpolation operators. The spectral radius of  $M$  with SBP-preserving interpolators is less than the one with the conventional prolongation and restriction.

stability limit. This phenomenon can be evaluated by looking at the eigenvalues on the left side of the picture: increasing the time-step  $\Delta t$ , the magnitude of these eigenvalues becomes bigger.

These numerical experiments suggest that the SBP-preserving interpolation operators yield multigrid iteration schemes with significantly improved convergence properties.

## 5.2 HIGHER ORDER METHODS

The following tests are carried out by raising the order of accuracy of the fine grid first-derivative operator. The tolerance  $\varepsilon = 10^{-10}$  was used.

|                        | Minimum iterations | $\Delta t$          | $\rho(M)$ | $\Delta t^*$        |
|------------------------|--------------------|---------------------|-----------|---------------------|
| (4, 2), SBP-preserving | 1435               | $5.8 \cdot 10^{-4}$ | 0.96882   | $5.9 \cdot 10^{-4}$ |
| (4, 2), Conventional   | 6358               | $1.3 \cdot 10^{-4}$ | 0.99404   | $3.4 \cdot 10^{-4}$ |
| (6, 3), SBP-preserving | 1278               | $5.1 \cdot 10^{-4}$ | 0.96955   | $6.8 \cdot 10^{-4}$ |
| (6, 3), Conventional   | 7030               | $9.8 \cdot 10^{-5}$ | 0.99485   | $2.6 \cdot 10^{-4}$ |
| (8, 4), SBP-preserving | 1856               | $6.8 \cdot 10^{-4}$ | 0.98502   | $7.2 \cdot 10^{-4}$ |
| (8, 4), Conventional   | 10817              | $7.9 \cdot 10^{-5}$ | 0.99667   | $2.3 \cdot 10^{-4}$ |

Table 2: The minimum number of iterations and the corresponding smoothing step are presented for each order of accuracy. For the SBP-preserving choice the minimum amount of iterations is almost the same for any order. The spectral radius  $\rho(M)$  and the time-step limit  $\Delta t^*$  are also given for each scheme.

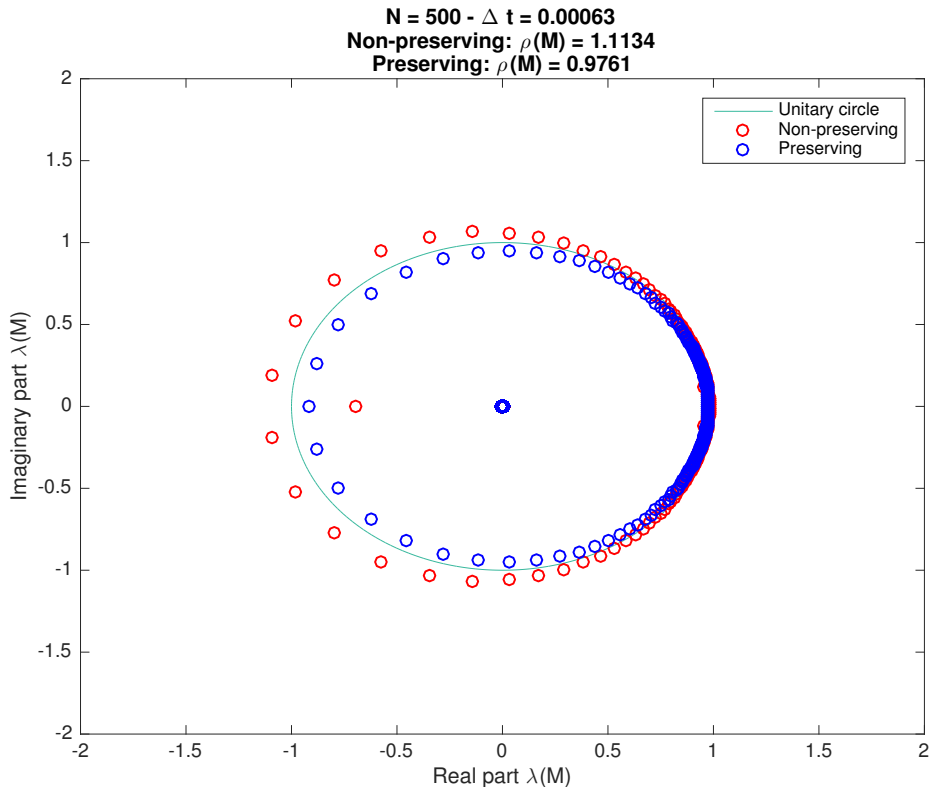


Figure 3: Increasing the smoothing step to  $\Delta t = 6.3 \cdot 10^{-4}$  leads to an unstable multigrid iteration scheme, if the non-preserving choice is made.

In Table 2 the effects of the interpolation operators on the multigrid iteration scheme are shown. For each order of accuracy, the SBP-preserving interpolators lead to faster convergence. Moreover, the minimum amount of iterations does not change considerably for higher orders. Conversely, when the order of accuracy of the fine-grid operator is increased and the classical linear interpolation (9) and (10) is used, the number of iterations grows and the time-step limit  $\Delta t^*$  decreases.

## 6 CONCLUSIONS

A new multigrid scheme for SBP-SAT discretizations has been proposed. So called SBP-preserving interpolation operators have been used to convey information between grids. When the Galerkin condition is considered, these prolongation and restriction operators lead to coarse-grid operators which satisfy the summation-by-part property.

Numerical experiments show that when compared to a multigrid scheme with the conventional choice of prolongation and restriction based on linear interpolation, the new scheme allows the use of larger smoothing steps and reaches convergence in fewer iterations. Hence, for every order of accuracy the SBP-preserving interpolation operators lead to multigrid iteration schemes which are more stable and have better convergence properties.

## REFERENCES

[1] U. Trottenberg, C.W. Oosterlee, A. Schuller, *Multigrid*, Academic Press, 2000.

- [2] W. L. Briggs, V. E. Henson, S. F. McCormick, *A Multigrid Tutorial*, Second Edition, SIAM, 2000.
- [3] C. Stolk, M. Ahmed, S. K. Bhowmik, *A Multigrid Method for the Helmholtz Equation with Optimized Coarse Grid Corrections*, SIAM Journal on Scientific Computing, Volume 36, Issue 6, 2014.
- [4] J. H. Adler et al., *Monolithic Multigrid Methods for Two-Dimensional Resistive Magnetohydrodynamics*, SIAM Journal on Scientific Computing, Volume 38, pp. B1-B24, 2016.
- [5] C. D. Santiago, C. H. Marchi, L. F. Souza, *Performance of geometric multigrid method for coupled two-dimensional systems in CFD*, Applied Mathematical Modelling, Volume 39, Issue 9, pp. 2602-2616, 2015.
- [6] K. Mattson, M. H. Carpenter, *Stable and Accurate Interpolation Operators for High-Order Multi-Block Finite-Difference Methods*, SIAM Journal on Scientific Computing, Volume 32, Issue 4, 2010.
- [7] M. Svärd, J. Nordström, *Review of Summation-By-Parts Schemes for Initial-Boundary-Value Problems*, Journal of Computational Physics, Volume 268, pp. 17-38, 2014.
- [8] A. Quarteroni, A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Berlin Heidelberg, pp.65-71, 2008.
- [9] L. Hogben, *Handbook of Linear Algebra*, CRC Press, p. 2-12, 2007.
- [10] B. Strand, *Summation by Parts for Finite Difference Approximations for  $d/dx$* , Journal of Computational Physics, Volume 110, pp. 47-67, 1994.
- [11] M. H. Carpenter, D. Gottlieb, S. Abarbanel, *Time-stable boundary conditions for finite-difference schemes solving hyperbolic systems: Methodology and application to high-order compact schemes*, Journal of Computational Physics, Volume 111 No. 2, pp. 220-236, 1994.
- [12] B. Gustafsson, P. Lötstedt, *Proceedings of the Fourth Copper Mountain Conference on Multigrid Methods*, pp.181-233, SIAM, Philadelphia, 1989.