ECCOMAS Congress 2016
VII European Congress on Computational Methods in Applied Sciences and Engineering
M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.)
Crete Island, Greece, 5–10 June 2016

SHAKEDOWN ANALYSIS OF PLATE BENDING UNDER STOCHASTIC UNCERTAINITY BY CHANCE CONSTRAINT PROGRAMMING

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Keywords: Limit analysis, Shakedown analysis, stochastic programming, chance constrained programming, non-linear programming.

Abstract. *In this paper we propose a stochastic programming to analyze limit and shakedown of plate bending under uncertainty conditions of strength. The Kirchhoff plate theory is used to formulate chance constrained problems. Based on the duality theory, the shakedown load multiplier formulated by the kinematic theorem is proved actually to be the dual form of the shakedown load multiplier formulated by static theorem. In this investigation a dual chance constrained programming algorithm is developed to calculate simultaneously both the upper and lower bounds of the plastic collapse limit and the shakedown limit.*

1 INTRODUCTION

An essential concern in mechanical and civil engineering design is to determine the ultimate load bearing capacity of structures beyond the elastic limit. Structures are considered safe if a state is reached after initial plastic deformation, such that the system does neither fail due to instantaneous or incremental collapse nor due to alternating plasticity. Using the classical step-by-step methods for such calculations can be cumbersome and computationally expensive in many cases. Furthermore, the disadvantage of these methods is that the exact knowledge of the loading history is necessary, which is not realistic in many technical applications. Limit and shakedown analysis are appropriate direct methods to avoid these problems. Reliability analysis of the structures takes the uncertainty of the actual load-carrying capacity of the structure into consideration since all resistance and loading variables are random in nature [1, 9, 10, 20].

Chance constrained programing is an approach of stochastic programming to limit and shakedown analysis under uncertainty [1, 2]. Under uncertainty the shakedown problem can be stated with random objective function or with random constraints. A probability is set with which the constraint has to be satisfied. Deterministic limit analysis of plates subjected to bending has been studied analytically and numerically. Both the Kirchhoff plate and the Mindlin plate models are considered based on the kinematic theorem of limit analysis and the finite element method. Based on the duality theory, Tran [3] developed a dual algorithm to calculate simultaneously both the upper and lower bounds of the plastic collapse limit and shakedown load of plates. In this study, we present a new primal-dual numerical algorithm of the shakedown problem under uncertainty in which the material strength is considered as a normally distributed random variable. Using chance constrained programming, we obtain deterministic equivalent formulations based on upper bound and lower bound theorems and then prove that formulations are actually dual to each other. The four-node discrete Kirchhoff quadrilateral (DKQ) element is used to analyze the problem. The proposed algorithm shows good performance in numerical test examples.

2 SHAKEDOWN ANALYSIS OF PLATE BENDING

2.1 Basic relations

Similar to the elastic theory, the inelastic behavior of thin plates may be analyzed under Kirchhoff's assumption that the normal to the middle plane of the plate remains straight and normal to the deformed middle plane. This assumption yields $u=-z\dfrac{\partial w}{\partial x}$, $v=-z\dfrac{\partial w}{\partial y}$, in which x, y are the coordinates in the middle plane of the plate, z is the distance from the middle plane, w is the deflection, u and v are the displacements in the x and y directions respectively. By differentiation, the strains are obtained as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y} \end{Bmatrix} = \begin{Bmatrix} -z\dfrac{\partial^2 w}{\partial x^2} \\ -z\dfrac{\partial^2 w}{\partial y^2} \\ -2z\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = z\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{1}$$

The vector $\boldsymbol{\chi} = \left[\chi_x \;\; \chi_y \;\; \chi_{xy} \right]^T$ is called the vector of curvatures.

The kinematic relations can be written as follows:

$$\dot{\boldsymbol{\chi}} = -\nabla^2 \dot{w} \tag{2}$$

where $\dot{\boldsymbol{\chi}}$ is the curvature rate vector and $\dot{w}$ is the transversal velocity.

In this paper, the von Mises yield criterion is considered. Expressing in terms of moments, the criterion takes the form $f(m_x, m_y, m_{xy}) = \left(m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 \right)^{1/2} - m_0 = 0$.

In matrix form the von Mises yield criterion can be written as follows:

$$f(\mathbf{m}) = \sqrt{\mathbf{m}^{\mathrm{T}} \mathbf{P} \mathbf{m}} - m_0 = 0 \tag{3}$$

where $\mathbf{m} = \left[m_x, m_y, m_{xy} \right]^{\mathrm{T}}$ is the vector of bending and twisting moments, $m_0 = \sigma_y t^2 / 4$ is the plastic limit moment per unit length of a plate section, t is the thickness of the plate, σ_y is the yield stress of the material, and

$$\mathbf{P} = \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}. \tag{4}$$

2.2 Static formulation

Consider a convex polyhedral load domain $\mathcal{L}$ and a special loading path consisting of all load vertices $\hat{P}_k (k = 1, ..., m) \in \mathcal{L}$. Let a point on the problem domain A (the midplane) be identified by a vector variable $\mathbf{x}$ and the fictitious elastic moment vector be $\mathbf{m}^{\mathrm{E}}$. The static shakedown theorem states that shakedown occurs if, and only if, there exists a time-independent self-equilibrium residual moment field $\overline{\boldsymbol{\rho}}$ which is statically admisible so that the actual moment field $\mathbf{m} = \alpha \mathbf{m}^{\mathrm{E}} + \overline{\boldsymbol{\rho}}$ does not anywhere violate the yield condition at any point of the structure and for all posible load combinations. Based on this theorem, we can find a statically admisible residual moment field in order to obtain a maximum load domain $\alpha \mathcal{L}$ that guarantees (3). Therefore, the shakedown load factor α^- can be obtained by solving the following optimization problem:

$$\begin{aligned} & \alpha^- = \max \alpha \\ & s.t.: \begin{cases} \nabla^2 \overline{\boldsymbol{\rho}}(\mathbf{x}) = 0 & \text{in } A \\ f(\alpha \mathbf{m}^{\mathrm{E}} + \overline{\boldsymbol{\rho}}) \le m_0 \end{cases} \end{aligned} \tag{5}$$

By discretizing the entire problem domain into finite elements and applying the Gauss-Legendre integration technique, Eqs (5) can be rewritten in the following form:

$$\alpha^{-} = \max \alpha$$

$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 & \text{in } A \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \le m_0 & \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \end{cases} \tag{6}$$

in which $\mathbf{B}_i$ is the deformation matrix, w_i is integration weight at Gauss point i; NG denotes the total number of Gauss points of the structure, and m is the number of load vetices..

2.3 Kinematic formulation

An upper bound to the shakedown limit of plate can be obtained using the kinematic shakedown theorem which has the following two statements:

(a) *Shakedown will occur for a structure subject to repeated or cyclic loads, if the plastic dissipation power exceeds the work rate of external forces for any admissible plastic strain-rate cycles and all loading paths*

(b) *Shakedown cannot occur, if the plastic dissipation power is less than the work rate of external forces for any one admissible plastic strain-rate cycle or any one loading path.*

We introduce here an admissible cycle of plastic curvature field $\Delta\boldsymbol{\chi}^p$. At each load vertex, the plastic curvature rate may not necessarily be compatible at each instant during the time cycle, but the plastic curvature accumulation over the cycle is required to be kinematically compatible such that

$$\Delta\boldsymbol{\chi}^p = \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_k^p = \nabla^2 \dot{w} \,. \tag{7}$$

Based on the above statements and the mathematical programming theory, an upper bound of the shakedown load factor can be found by solving the following convex nonlinear programming (the superscript p is neglected for simplicity):

$$\alpha^{+} = \min \sum_{k=1}^{m} \int_A D_{\text{int}}(\dot{\boldsymbol{\chi}}_k)\, dA$$

$$\text{s.t.:} \begin{cases} \Delta\boldsymbol{\chi} = \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_k = \nabla^2 \dot{w} & \text{in } A \\ \dot{w} = 0 & on \ \partial A \\ \sum_{k=1}^{m} \int_A \mathbf{m}^E(x, \hat{P}_k)^T \dot{\boldsymbol{\chi}}_k dA = 1 \end{cases} \tag{8}$$

where $D_{\text{int}}(\boldsymbol{\chi})$ is the plastic dissipation power per unit area. According to the von Mises yield condition, the plastic dissipation can be expressed in term of the curvature rates of the middle surface: $D_{\text{int}}(\dot{\boldsymbol{\chi}}) = m_0 \sqrt{\dot{\boldsymbol{\chi}}_k{}^T \mathbf{Q} \dot{\boldsymbol{\chi}}_k}$

with

$$\mathbf{Q} = \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

We denote the nodal variables of the finite element by $\mathbf{q} = \left[w \quad \partial w/\partial x \quad \partial w/\partial y\right]^{\mathrm{T}}$, the discretized formulation by FEM is as follows:

$$\alpha^{+} = \min \sum_{k=1}^{m}\sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}{}^{T}\mathbf{Q}\dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:}\begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{q}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m}\sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{T} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{10}$$

3 STOCHASTIC SHAKEDOWN ANALYSIS OF PLATE BENDING BY CHANCE CONSTRAINED PROGRAMMING

3.1 Static Approach to chance constrained programming

Consider the situation that the strength of material is not given, fixed quantity, but must be modelled $R = R(\omega)$ through random variables on a certain probability space. Hence, the shakedown problem (6),(10), respectively, under stochastic uncertainty must be reformulated by appropriate deterministic problems which are provided by adopting stochastic optimization techniques. A technique used effectively called chance constrained programming is presented here.

For stochastic plastic moment $\mathbf{m}_i(\omega)$ depending on some random ω, a stochastic formulation can be obtained by assuming that the inequality constraints of (6) are satisfied at least by a chance $\psi = \left[0.5\,;\,1\right]$ (or reliability level) at Gauss point i.

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^{T} \overline{\boldsymbol{\rho}}_i = 0 & \text{in } A \\ \text{Prob}\left[f(\alpha \mathbf{m}_{ik}^{E} + \overline{\boldsymbol{\rho}}_i) - m_{0i}(\omega) \le 0 \right] \ge \psi & \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \end{cases} \tag{11}$$

Based on the methodology of chance constrained programming, the stochastic program (11) is to be relaxed into an equivalent deterministic optimization problem:

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^{T} \overline{\boldsymbol{\rho}}_i = 0 & \text{in } A \\ f(\alpha \mathbf{m}_{ik}^{E} + \overline{\boldsymbol{\rho}}_i) \le \mu_i - \kappa_i \sigma_i & \forall i = \overline{1, NG}, \quad \forall k = \overline{1, m} \end{cases} \tag{12}$$

where $\mu_i, \sigma_i, \kappa_i = \Phi^{-1}(\psi_i)$ is the mean value, standard deviation and the inverse of the cumulative distribution function of plastic moment at Gauss point i, respectively.

3.2 Kinematic approach to chance constrained programming

As mentioned above the deterministic formulation to calculate an upper bound shakedown load factor:

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}{}^T \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$

$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{q}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^T \mathbf{m}_{ik}^E = 1 \end{cases} \tag{13}$$

If the yield stress of material is random, then the plastic moment is uncertainty quantity and the objective function of (13) is a stochastic variable. Firstly, we must properly define the minimum of a random function. This can be done in such a way that one looks for a minimum lower bound η objective function under the constraint that the probability of violation of that bound is prescribed [4].

$$\min \eta$$

$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0(\omega) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}} \geq \eta \right) = \psi \\ \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{q}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^E = 1 \end{cases} \tag{14}$$

The chance constrained program technique also can convert the stochastic program (14) into an equivalent deterministic program as follows:

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \left(\mu_i - \kappa_i \sigma_i \right) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$

$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{q}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^E = 1 \end{cases} \tag{15}$$

4 DUALITY APPROACH TO CHANCE CONSTRAINED PROGRAMMING

4.1 Dual theorem for chance constrained programming

Some new notations is intruduced for the sake of simplicity:

$$\dot{\mathbf{k}}_{ik} = w_i \mathbf{Q}^{1/2} \dot{\boldsymbol{\chi}}_{ik}, \qquad \mathbf{t}_{ik} = \left(\mathbf{Q}^{-1/2}\right)^T \mathbf{m}_{ik}^E, \qquad \hat{\mathbf{B}}_i = w_i \mathbf{Q}^{1/2} \mathbf{B}_i \tag{16}$$

where

$$\mathbf{Q}^{1/2}\mathbf{Q}^{-1/2} = \mathbf{I}, \;\; \mathbf{Q} = \left(\mathbf{Q}^{1/2}\right)^T \mathbf{Q}^{1/2} \tag{17}$$

By substituting (16) into (15) one obtains a simplified version for upper bound of shakedown limit load (primal problem)

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} (\mu_i - \kappa_i \sigma_i) \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}$$

$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{q}} = \mathbf{0} \qquad \forall i = \overline{1, NG} & \text{(a)} \\ \sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 & \text{(b)} \end{cases} \tag{18}$$

It is seen that in the case of limit analysis there exists a dual form for (18), see, e.g. Heitzer and Staat [16] , Andersen *et al.* [17]. Vu *et al*. [18], [19] generalized this theory for the case of shakedown analysis. An extension of their theory to shakedown analysis of plate is investigated in this work through the following theorem:

> **Theorem**. *If there exists a finite solution α^+ for the upper bound shakedown limit load multiplier (20) then the static formulation (12) is exactly the dual problem of the kinematic one (20) such that*

$$\alpha^+ = \min_{\mathbf{h}\left(\dot{\mathbf{k}}_{ik}, \dot{\mathbf{q}}\right) = \mathbf{0}} \sum_{k=1}^{m} \sum_{i=1}^{NG} (m_i - \kappa_i \sigma_i) \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}} = \max_{\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left(\alpha \mathbf{m}_{ik}^E + \bar{\boldsymbol{\rho}}_i\right) \le 0 \end{cases}} \alpha = \alpha^- \tag{19}$$

4.2 Primal-dual shakedown algorithm

A difficult happening when dealing with the non-linear constrained optimization problem (20) is that the objective function is not everywhere differentiable. In order to allow a direct non-linear, smooth optimization problem, a ‘smooth regularization method’ should be used for overcoming this barrier. For this purpose, a very small positive number, ε_0^2 will be added to $D_{\text{int}}(\dot{\boldsymbol{\chi}}_k)$. An efficient technique for large-scale optimization problems, which are successfully applied in [13] is used. Using penalty method to eliminate the first constraint in (18) lead to a penalty function

$$P = \sum_{i=1}^{NG} \left\{ \sum_{k=1}^{M} \left((m_i - \kappa_i \sigma_i) \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2} \right) + \frac{c}{2} \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{q}} \right)^T \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{q}} \right) \right\} \tag{20}$$

where c is a penalty parameter such that $c \gg 1$. The corresponding Lagrangian of (20) is

$$L = P - \alpha \left(\sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^T \mathbf{t}_{ik} - 1 \right) \tag{21}$$

We denote

$$\boldsymbol{\beta}_i = -c\left(\sum_{k=1}^{m}\dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{q}}\right) \tag{22}$$

By employing Newton method to solve the KKT conditions of the Lagrangian in (39) and after some manipulations, one gets the following system:

$$\mathbf{K}d\dot{\mathbf{q}} = -\mathbf{K}\dot{\mathbf{q}} + \mathbf{f}_1 + \mathbf{f}_2(\alpha + \mathrm{d}\alpha) \tag{23}$$

in which

$$\begin{aligned}
\mathbf{K} &= \sum_{i=1}^{NG}\hat{\mathbf{B}}_i^T\mathbf{E}_i^{-1}\hat{\mathbf{B}}_i \\
\mathbf{f}_1 &= -\sum_{i=1}^{NG}\hat{\mathbf{B}}_i^T\mathbf{E}_i^{-1}\sum_{k=1}^{m}\mathbf{M}_{ik}^{-1}(\boldsymbol{\beta}_i + \alpha\mathbf{t}_{ik})\frac{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}} \\
\mathbf{f}_2 &= \sum_{i=1}^{NG}\hat{\mathbf{B}}_i^T\mathbf{E}_i^{-1}\sum_{k}^{m}\mathbf{M}_{ik}^{-1}\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\mathbf{t}_{ik}
\end{aligned} \tag{24}$$

Solving the system (23), we have the incremental vectors of nodal variables $\dot{\mathbf{q}}$, curvature rate $\dot{\mathbf{k}}_{ik}$ and $\boldsymbol{\beta}_i$ as follows :

$$\begin{aligned}
d\dot{\mathbf{q}} &= d\dot{\mathbf{q}}_1 + d\dot{\mathbf{q}}_2\left(\alpha + d\alpha\right) \\
d\dot{\mathbf{k}}_{ik} &= (d\dot{\mathbf{k}}_{ik})_1 + (d\dot{\mathbf{k}}_{ik})_2\left(\alpha + d\alpha\right) \\
d\boldsymbol{\beta}_i &= (d\boldsymbol{\beta}_i)_1 + (d\boldsymbol{\beta}_i)_2\left(\alpha + d\alpha\right)
\end{aligned} \tag{25}$$

and

$$\left(\alpha + d\alpha\right) = \frac{1 - \sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\left[\dot{\mathbf{k}}_{ik} + \left(d\dot{\mathbf{k}}_{ik}\right)_1\right]}{\sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\left(d\dot{\mathbf{k}}_{ik}\right)_2} \tag{26}$$

The vectors $\mathrm{d}\dot{\mathbf{q}}, \mathrm{d}\dot{\mathbf{k}}_{ik}, \mathrm{d}\boldsymbol{\beta}_i$ and $\mathrm{d}\alpha$ are actually Newton directions which assure that a suitable step along them will lead to a decrease of the objective function of the primal problem (18) and to an increase of the objective function of the objective function of the dual problem (12). Based on(25), (26) we can update the vectors of $\dot{\mathbf{q}}, \dot{\mathbf{k}}_{ik}$, $\boldsymbol{\beta}_i$ and α .

5 NUMERICAL EXAMPLES

In this section, the numerical solution of some problems is presented to test the performance of the dual shakedown algorithm. Plates subjected to uniform or concentrated loads are considered. The 4-node DKQ plate element is applied for structural discretization. For all examples the following was assumed: length $L = 10m$, plate thickness $t = 0.1\text{m}$, the mean value of yield stress $\mathrm{E}(\sigma_y) = 250\,\mathrm{MPa}$ and the standard deviation $\sigma = 0.1\mathrm{E}(\sigma_y)$. A reliability level $\psi = 0.9999$ is chosen.

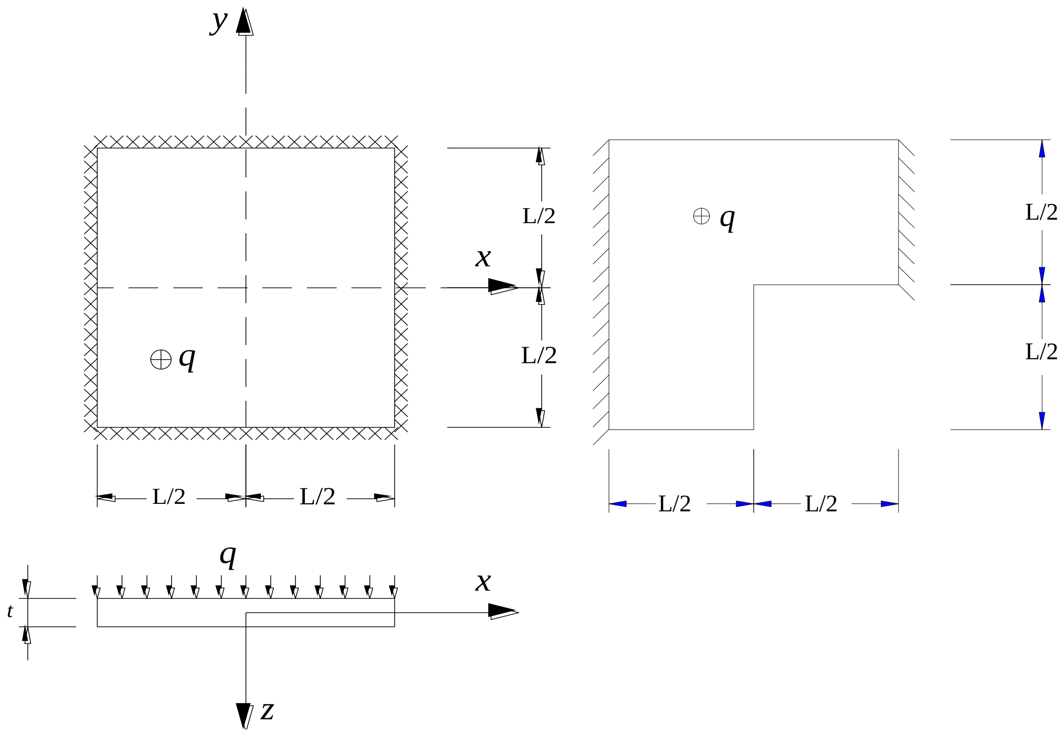

Figure 1. Square plate and L-shape plate loaded by a uniform pressure

5.1 Limit analysis of square plate subjected to uniform load

Firstly, we consider a square plate subjected to uniform pressure q as shown on Figure 1. In this analysis, the plate is modelled by 256 DKQ elements due to symmetry. Table 1 shows the comparison of the present numerical results for both cases, simply supported and clamped plates.

Authors	Upper/Lower bound (deterministic)		Upper/Lower bound (random strength)	
	Spl. supported	Clamped	Spl. supported	Clamped
Hodge *et al.* [14]	26.54/24.86	49.25/42.86	–/–	–/–
Lubliner [17]	27.71/23.81	52.01/–	–/–	–/–
Capsoni *et al.* [15]	25.02/–	45.29/–	–/–	–/–
Le *et al.* [16]	25.01/–	45.29/–	–/–	–/–
Present	25.04/25.04	45.06/45.06	15.72/15.72	28.36/28.36

Table 1. Limit load factor of square plates in comparison with other solutions $\left(\dfrac{m_0}{qL^2}\right)$.

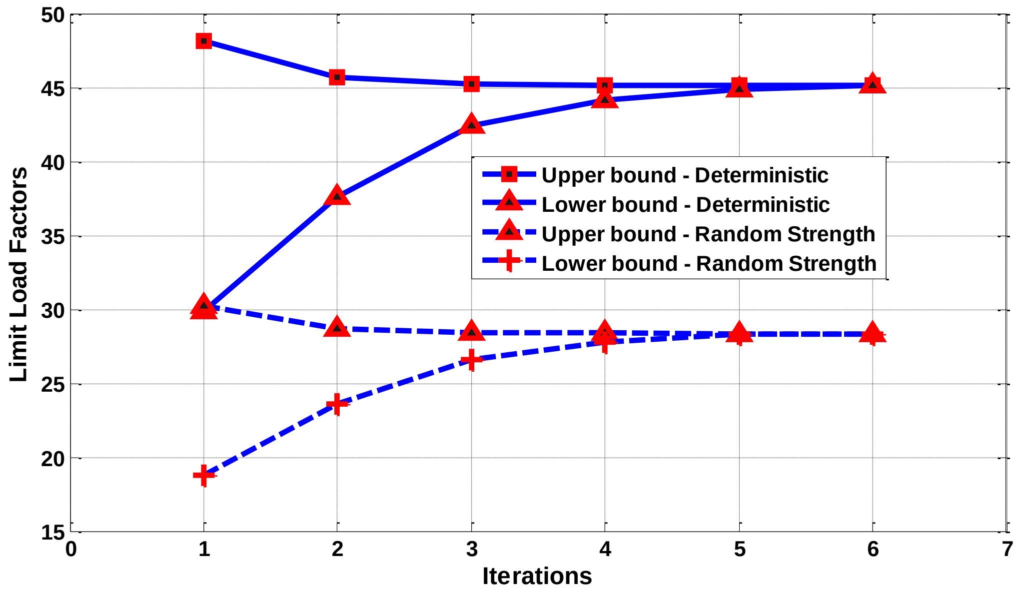

Figure 2. Clamped square plate: Convergence of limit load factors.

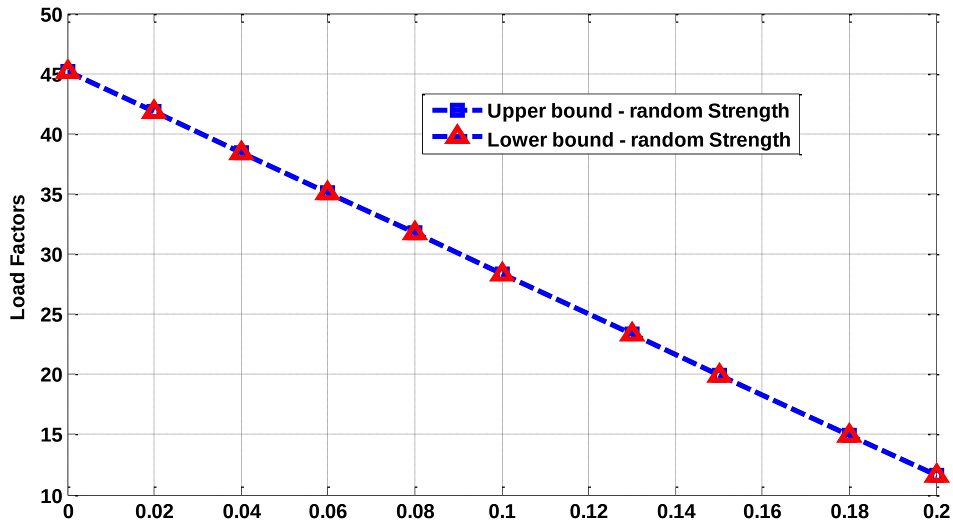

Figure 3. Dependence of load factors on the coefficient of variation $\xi\left(\xi = \frac{\sigma}{E}\right)$.

5.2 L-shaped plate subjected to uniform load.

In the second example, we investigate an L-shape plate subjected to uniform pressure q (Figure 1) which can be constant or vary within a range $q \in [0 \;\; q_{\max}]$. In this analysis, the plate is modelled by 768 DKQ elements. Figure 4 and figure 5 show the convergence of the upper bound and lower bounds for the simply supported case.

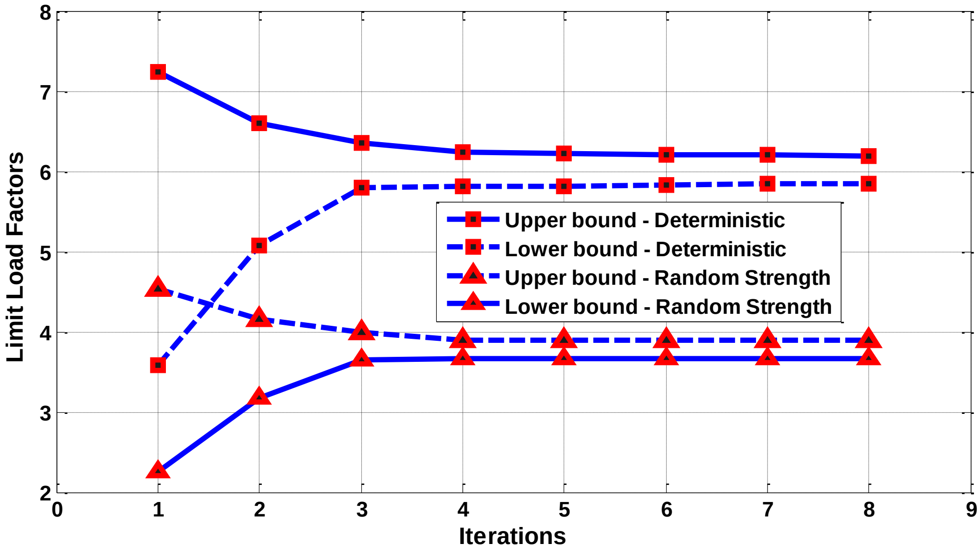

Figure 4: L-shape Plate: Convergence of limit load factors.

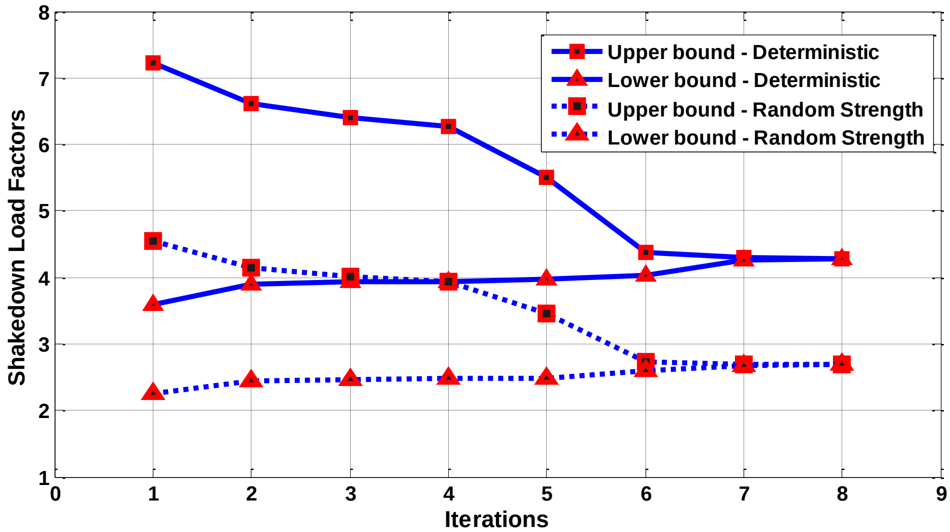

Figure 5: L-shape Plate: Convergence of shakedown load factors

Authors	Upper/Lower bound (deterministic)		Upper/Lower bound (random strength)	
	Limit	Shakedown	Limit	Shakedown
Le *et al.* [16]	6.289/–	–/–	–/–	–/–
Present	6.19/5.85	4.28/4.28	3.89/3.67	2.69/2.69

Table 2. Limit load factor of plate in comparison with other solutions $\left(\frac{m_0}{qL^2}\right)$

6 CONCLUSIONS

- Direct structural reliability design can be achieved by chance constraint programming.
- In the general case chance constraint programming is a hard problem because probabilities have to be calculated as high dimensional integrals during in the optimization algorithm.
- For normally distributed stochastic variables deterministic equivalents can be formulated (for linear programming).
- Extension to nonlinear programming is on its way.

For engineering design:

- Structural reliability is post design
- Stochastic programming is (pre) design
- The load factor decreases "quickly" with increasing coefficient of variation of the strength.
- High structural reliability can be achieved with moderate reduction of the load factor

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